# STIEFEL-WHITNEY CLASSES
# I. AXIOMS AND CONSEQUENCES

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ABSTRACT. After a brief review of cohomology theory we define the Stiefel-Whitney classes associated to a vector bundle and prove some consequences from their axioms. We proceed to compute the Stiefel-Whitney classes of projective space and apply the result to show non-existence of real division algebras in most dimensions.

**Notation.** We denote by $\epsilon^n$ the trivial $n$-dimensional vector bundle over a given space. Isomorphism in the respective category is denoted by $\cong$ (e.g. homeomorphism for topological spaces, isomorphism of Abelian groups, equivalence of bundles over a fixed base space).

**Acknowledgements.** Sections 2 and 3 are based on the presentation in [MS74, §4].

## 1. Cohomology Theory

Since the Stiefel-Whitney classes of a vector bundle are invariants which live in the cohomology groups of the base space we shall give a brief review of cohomology theory (cf. [Hat02], [Lü05] or [May99] for more detailed accounts).

**Axioms.** A *cohomology theory* with coefficients in an $R$-module $M$ is a contravariant functor

$$H^*(\cdot; M) : \text{Topological pairs} \to \mathbb{Z}\text{-graded Abelian groups}$$

together with natural transformations

$$\partial : H^k(A; M) \to H^{k+1}(X, A; M)$$

satisfying the *Eilenberg-Steenrod axioms*[1]:

(C1) homotopy-invariance: any two homotopic maps induce the same morphism

(C2) long exact sequence: any pair $(X, A)$ induces a long exact sequence of the form

$$\cdots \longrightarrow H^k(X, A; M) \xrightarrow{\text{id}^*} H^k(X; M) \xrightarrow{\text{incl}^*} H^k(A; M) \xrightarrow{\partial} H^{k+1}(X, A; M) \longrightarrow \cdots$$

(C3) excision: given subspaces $\text{cl}(Z) \subseteq \text{int}(A) \subseteq X$ we have induced isomorphisms

$$H^k(X, A; M) \xrightarrow[\cong]{\text{incl}^*} H^k(X \backslash Z, A \backslash Z; M)$$

[1]We write $X$ for a pair $(X, \varnothing)$ and we denote by $f^*$ the morphism $H^k(f)$ induced by a map $f$.

(C4) product axiom: given a family of topological pairs $(X_i, A_i)$ we have induced isomorphisms

$$H^k(\coprod_i (X_i, A_i); M) \xrightarrow{\prod_i \mathrm{incl}_i^*} \prod_i H^k(X_i, A_i; M)$$

An *ordinary cohomology theory* also satisfies the following axiom:

(C5) dimension axiom:

$$H^k(\{*\}; M) = \begin{cases} M & , k = 0 \\ 0 & , k \neq 0 \end{cases}$$

**Theorem 1.** *There exists a cohomology theory for an arbitary coefficient module, called* singular cohomology*.*

**Reduced Cohomology.** For calculations it is often a nuisance that the cohomology groups of a point are trivial. This motivates the definition of the *reduced cohomology groups*

$$\tilde{H}^k(X; M) := H^k(X, \{*\}; M)$$

**Proposition 2.**

$$H^k(X; M) \cong \tilde{H}^k(X; M) \oplus H^k(\{*\}; M)$$

*In particular, points (and hence contractible spaces) have trivial reduced cohomology.*

*Proof.* By the long exact sequence (C2)

$$\cdots \longrightarrow \tilde{H}^k(X; M) \xrightarrow{\mathrm{id}^*} H^k(X; M) \xrightarrow{\mathrm{incl}^*} H^k(\{*\}; M) \xrightarrow{\partial} \tilde{H}^{k+1}(X; M) \longrightarrow \cdots$$

Now, every space retracts to a point, i.e. $\mathrm{const}_* \circ \mathrm{incl} = \mathrm{id}_{\{*\}}$. Hence $\mathrm{incl}^* \circ \mathrm{const}_*^* = \mathrm{id}_{H^k}$ and we have a split exact sequence

$$0 \longrightarrow \tilde{H}^k(X; M) \xrightarrow{\mathrm{id}^*} H^k(X; M) \underset{\mathrm{const}_*^*}{\overset{\mathrm{incl}^*}{\rightleftarrows}} H^k(\{*\}; M) \longrightarrow 0$$

□

There is also a long exact sequence for reduced cohomology:

**Theorem 3.** *Any pair $(X, A)$ induces a long exact sequence of the form*

$$\cdots \longrightarrow H^k(X, A; M) \longrightarrow \tilde{H}^k(X; M) \longrightarrow \tilde{H}^k(A; M) \longrightarrow H^{k+1}(X, A; M) \longrightarrow \cdots$$

*(Observe that the base point $*$ has to be chosen in $A$.)*

*Proof.* By functoriality, naturality and the long exact sequence (C2) we have the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
\cdots \dashrightarrow & H^k(X, A; M) & \dashrightarrow & \tilde{H}^k(X; M) & \dashrightarrow & \tilde{H}^k(A; M) & \dashrightarrow & H^{k+1}(X, A; M) & \dashrightarrow \cdots \\
& \downarrow \mathrm{id} & & \downarrow \mathrm{id}^* & & \downarrow \mathrm{id}^* & & \downarrow \mathrm{id} & \\
\cdots \longrightarrow & H^k(X, A; M) & \xrightarrow{\mathrm{id}^*} & H^k(X; M) & \xrightarrow{\mathrm{incl}^*} & H^k(A; M) & \xrightarrow{\partial} & H^{k+1}(X, A; M) & \longrightarrow \cdots \\
& \downarrow \mathrm{incl}^*=0 & & \downarrow \mathrm{incl}^* & & \downarrow \mathrm{incl}^* & & \downarrow \mathrm{incl}^*=0 & \\
\cdots \longrightarrow & H^k(\{*\}, \{*\}; M) & \xrightarrow{\mathrm{id}^*} & H^k(\{*\}; M) & \xrightarrow{\mathrm{incl}^*} & H^k(\{*\}; M) & \xrightarrow{\partial} & H^{k+1}(\{*\}, \{*\}; M) & \longrightarrow \cdots
\end{array}$$

The existence of the dotted maps in the first row follows from diagram chasing. □

**Spheres.** Long exact sequences are the main tool in computing the cohomology of spaces.

**Example 4.** The reduced cohomology groups of the spheres are given by

$$\tilde{H}^k(S^n; M) = \begin{cases} M & , n = k \\ 0 & , \text{otherwise} \end{cases}$$

*Proof.* (1) The assertion for $S^0 = \{\pm 1\}$ is immediate from the product and dimension axioms (C4) and (C5).

(2) On the other hand it follows from the long exact reduced cohomology sequence for the pair $(D^{n+1}, S^n)$ that

$$\tilde{H}^k(S^n; M) \cong H^{k+1}(D^{n+1}, S^n; M)$$

(3) Now consider the "wrap-around map" $f : (D^{n+1}, S^n) \to (S^{n+1}, \{*\})$ which is a homeomorphism away from the boundary (it is the inverse of stereographic projection from $*$). We define subspaces

$$S^n \subseteq \underbrace{(D^{n+1} \backslash \tfrac{3}{4} D^{n+1})}_{=:Z} \subseteq \underbrace{\tfrac{1}{2} D^{n+1}}_{=:A} \subseteq D^{n+1},$$
$$* \in f(Z) \subseteq f(A) \subseteq S^{n+1}$$

and get a commutative diagram

$$\begin{array}{ccc} (D^{n+1}, S^k) & \xrightarrow{f} & (S^{n+1}, \{*\}) \\ \downarrow \simeq & & \downarrow \simeq \\ (D^{n+1}, A) & \xrightarrow{f} & (S^{n+1}, f(A)) \\ \uparrow & & \uparrow \\ (D^{n+1} \backslash Z, A \backslash Z) & \xrightarrow[\cong]{f} & (S^{n+1} \backslash f(Z), f(A) \backslash f(Z)) \end{array}$$

The upper inclusions are homotopy equivalences, the lower inclusions satisfy the hypotheses of excision and the bottom restriction is a homeomorphism. Hence they induce isomorphism of the cohomology groups and it follows that $f$ induces the first isomorphism in

$$\tilde{H}^{k+1}(S^{n+1}; M) \cong H^{k+1}(D^{n+1}, S^n; M) \cong \tilde{H}^k(S^n; M)$$

The claim now follows by induction. □

**Multiplicative Structure.** A *multiplicative structure* on a cohomology theory $H^*(\cdot; M)$ with coefficients in an $R$-module $M$ is given by a family of $R$-linear maps

$$\cup : H^k(X, A; M) \times H^l(X, B; M) \to H^{k+l}(X, A \cup B)$$

(called the *cup product*) together with a unit $1_X \in H^0(X; M)$ satisfying the following axioms:

(M1) associativity

(M2) graded commutativity:

$$x \cup y = (-1)^{kl} y \cup x \qquad (\forall x \in H^k(X, A; M), y \in H^l(X, B; M))$$

(M3) naturality

(M4) compatibility with the boundary map: if $(X, A)$ is any pair, then

$$\partial(a) \cup x = \partial(a \cup \mathrm{incl}^*(x)) \qquad (\forall a \in H^k(A; M), x \in H^l(X; M))$$

We will usually omit the $\cup$ symbol from products. The product turns the groups

$$H^*(X, A; M) := \bigoplus_{k=0}^{\infty} H^k(X, A; M) \quad \text{and} \quad H^{**}(X, A; M) := \prod_{k=0}^{\infty} H^k(X, A; M)$$

into unital graded commutative rings, called the *cohomology rings.*

**Lemma 5.** *Elements of the form $1_X + x$ where the zeroth coefficient of $x$ vanishes are invertible in $H^{**}(X, A; M)$.*

**Theorem 6.** *Singular cohomology admits a multiplicative structure if $M = R$ is a commutative unital ring.*

**Projective Space.** The following theorem describes the structure of the singular cohomology ring of projective space with $\mathbb{Z}_2$-coefficients. Note that this is a commutative ring since $-1 = 1$ in $\mathbb{Z}_2$.

**Theorem 7.** *The cohomology groups of projective space with $\mathbb{Z}_2$-coefficients are given by*

$$H^k(P^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & , k \leqslant n \\ 0 & , \textit{otherwise} \end{cases}$$

*and the higher cohomology groups are generated by the respective powers of the generator $0 \neq a \in H^1(P^n; \mathbb{Z}_2)$.*

*That is, $H^*(P^n; \mathbb{Z}_2)$ is the unital graded commutative ring generated by a single element $a$ of degree 1 subject to the relation $a^{n+1} = 0$.*

## 2. Stiefel-Whitney Classes

**Axioms.** The *Stiefel-Whitney classes* are cohomology classes $w_k(\xi) \in H^k(X; \mathbb{Z}_2)$ assigned to each vector bundle $\xi : E \to X$ such that the following axioms are satisfied:

(S1) $w_0(\xi) = 1_X$

(S2) $w_k(\xi) = 0$ if $\xi$ is an $n$-dimensional vector bundle and $k > n$

(S3) naturality: $w_k(\xi) = f^*(w_k(\eta))$ if there is a bundle map $\xi \to \eta$ with base map $f$

(S4) Whitney product axiom: $w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi) w_j(\eta)$

(S5) non-triviality: $w_1(\gamma_1^1) \neq 0$ where $\gamma_1^1$ is the canonical line bundle over $P^1$

If we define the *total Stiefel-Whitney class* as $w(\xi) := (w_k(\xi)) \in H^{**}(X; \mathbb{Z}_2)$, then the fourth axiom can be equivalently stated as follows:

(S4') $w(\xi \oplus \eta) = w(\xi) w(\eta)$

Observe that every total Stiefel-Whitney class is invertible by axiom (S1) and Lemma 5. If $w(\xi) = 1$ then we say that the total Stiefel-Whitney class is *trivial.*

Finally we define the Stiefel-Whitney classes of a manifold $M$ in terms of its tangent bundle:

$$w_i(M) := w_i(\tau_M) \quad \text{and} \quad w(M) := w(\tau_M)$$

**Consequences.** The following properties follow directly from the axioms.

**Proposition 8.** *Two equivalent vector bundles have equal Stiefel-Whitney classes.*

*Proof.* A bundle equivalence is a bundle map inducing the identity as its base map; hence the claim follows by naturality (S3). □

**Proposition 9.** *Trivial vector bundles have trivial total Stiefel-Whitney class.*

*Proof.* Any trivial vector bundle over $X$ is equivalent to a vector bundle of the form $X \times \mathbb{R}^n$. Thus the diagram

$$\begin{array}{ccc} X \times \mathbb{R}^n & \xrightarrow{\text{proj}_2} & \mathbb{R}^n \\ {\scriptstyle \text{proj}_1}\downarrow & & \downarrow \\ X & \longrightarrow & \{*\} \end{array}$$

shows that there is a bundle map to a vector bundle over a point. But we know that all higher cohomology groups of a point are trivial. Hence the claim follows from naturality (S3). □

Recall that a manifold is called *parallelizable* if its tangent bundle is trivial. Thus the Stiefel-Whitney classes of a manifold measure obstruction to parallelizability:

**Corollary 10.** *Parallelizable manifolds have trivial Stiefel-Whitney class.*

**Corollary 11** (Stability)**.** *The Stiefel-Whitney classes do not change upon addition of a trivial bundle.*

*Proof.* Apply the Whitney product axiom (S4). □

**Corollary 12.** *Let $\xi$ be an $n$-dimensional Euclidean bundle. If $\xi$ possesses $k$ nowhere linearly dependent sections, then the Stiefel-Whitney classes $w_i(\xi)$ vanish already if $i > n - k$.*

*Proof.* The $k$ nowhere linearly dependent sections span a *trivial* $k$-dimensional subbundle $\eta \subseteq \xi$. By means of the continuous inner product we can construct a $(n-k)$-dimensional orthogonal complement $\eta^\perp$. We have $\eta \oplus \eta^\perp = \xi$, hence $w(\xi) = w(\eta^\perp)$ by stability, and the claim follows from axiom (S2). □

**Tangent and Normal Bundle.** Let $N \subseteq M$ be a smooth submanifold of a smooth Riemannian manifold. Then the tangent bundle of $N$ is a subbundle of the tangent bundle of $M$ restricted to $N$, i.e. $\tau_N \subseteq \tau_M|N$ and, as before, by means of the smooth inner product we can construct an orthogonal complement $\nu_N^M$ called the *normal bundle* of $N \subseteq M$, and $\tau_N \oplus \nu_N^M \cong \tau_M|N$.

**Corollary 13** (Whitney Duality Theorem)**.** *If $N \subseteq M$ is a smooth submanifold of a manifold in Euclidean space (i.e. $M \subseteq \mathbb{R}^n$ open with the identity chart), then*

$$w(N) = w(\nu_N^M)^{-1}$$

**Example 14.** The tangent bundle of a sphere has trivial total Stiefel-Whitney class.

*Proof.* The normal bundle for the standard embedding $S^n \subseteq \mathbb{R}^{n+1}$ is trivial. □

In particular, the tangent bundle of a sphere cannot be distinguished from a trivial bundle over the sphere by means of their Stiefel-Whitney classes.

## 3. Bundles over Projective Space

**Canonical Line Bundle.** We will now calculate the Stiefel-Whitney classes of the canonical line bundles directly from the axioms:

**Proposition 15.** *The total Stiefel-Whitney class of the canonical line bundle $\gamma_n^1$ over $P^n$ is given by*

$$w(\gamma_n^1) = 1 + a$$

*where $a$ denotes the generator of $H^*(P^n; \mathbb{Z}_2)$ (cf. Thm. 7).*

*Proof.* We have an obvious bundle map

$$\begin{array}{ccc} E(\gamma_1^1) & \longrightarrow & E(\gamma_n^1) \\ \downarrow \gamma_1^1 & & \downarrow \gamma_n^1 \\ P^1 & \xrightarrow[\text{incl}]{} & P^n \end{array}$$

Therefore

$$0 \overset{\text{(S5)}}{\neq} w_1(\gamma_1^1) = \text{incl}^*(w_1(\gamma_n^1))$$

and this shows that $w_1(\gamma_n^1) = a$, hence $w(\gamma_n^1) = 1 + a$ since the bundle is one-dimensional. □

**Tangent Bundle.** By definition the canonical line bundle over $P^n$ is a subbundle of the trivial $(n+1)$-dimensional bundle. We can thus consider its orthogonal complement $\gamma_n^\perp$ which is given by

$$E(\gamma_n^\perp) := \{([x], v) \in P^n \times \mathbb{R}^{n+1} : x \perp v\} \xrightarrow{\text{proj}_1} P^n$$

**Proposition 16.** *The total Stiefel-Whitney class of the orthogonal complement bundle is given by*

$$w(\gamma_n^\perp) = 1 + a + a^2 + \ldots + a^n$$

*Proof.* Since the Whitney sum $\gamma_n^1 \oplus \gamma_n^\perp = \epsilon^{n+1}$ is trivial (by construction) we have

$$w(\gamma_n^\perp) = w(\gamma_n^1)^{-1} = (1 + a)^{-1} = 1 + a + a^2 + \ldots + a^n$$

□

In particular, this shows that all of the first $n$ Stiefel-Whitney classes of an $n$-dimensional vector bundle may be non-zero.

**Proposition 17.** *The tangent bundle $\tau_{P^n}$ of $P^n$ is equivalent to the homomorphism bundle* $\text{Hom}(\gamma_n^1, \gamma_n^\perp)$.

*Proof.* Recall that the tangent bundle of projective space can be defined as follows:

$$TP^n := (TS^n := \{(x, v) \in S^n \times \mathbb{R}^{n+1} : x \perp v\})/\{\pm 1\} \longrightarrow S^n/\{\pm 1\} =: P^n$$

Note that every point of $S^n$ naturally represents a point in the canonical line bundle $\gamma_n^1$ and every point of the tangent plane of $S^n$ naturally represents a point in the orthogonal complement bundle $\gamma_n^\perp$. This suggest defining a map

$$TP^n \to \mathrm{Hom}(\gamma_n^1, \gamma_n^\perp), \ [(x, v)] \mapsto (x \mapsto v)$$

and it is straightforward to verify that all equivalence relations fit together in such a way that this map is a well-defined bundle equivalence. □

**Theorem 18.** *The following bundles are equivalent:*

$$\tau_{P^n} \oplus \epsilon^1 \cong \bigoplus_{k=1}^{n+1} \gamma_n^1$$

*In particular, the total Stiefel-Whitney class of projective space is given by*

$$w(P^n) = (1 + a)^{n+1}$$

*Proof.* The endomorphism bundle $\mathrm{Hom}(\gamma_n^1, \gamma_n^1)$ is trivial (consider the non-vanishing identity section). Thus

$$\tau_{P^n} \oplus \epsilon^1 \overset{17}{\cong} \mathrm{Hom}(\gamma_n^1, \gamma_n^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \cong \mathrm{Hom}(\gamma_n^1, \gamma_n^\perp \oplus \gamma_n^1)$$
$$\cong \mathrm{Hom}(\gamma_n^1, \epsilon^{n+1}) \cong \bigoplus_{k=1}^{n+1} \mathrm{Hom}(\gamma_n^1, \epsilon^1) \cong \bigoplus_{k=1}^{n+1} \gamma_n^1$$

where the last equivalence is induced by the continuous inner product of the Euclidean bundle $\gamma_n^1$. □

**Corollary 19** (Stiefel)**.** *The total Stiefel-Whitney class of the projective space $P^n$ is trivial if and only if $(n + 1)$ is a power of 2.*

*Proof.* Write $n + 1 = 2^k m$ with odd $m$. By the Frobenius homomorphism we have

$$w(P^n) = (1 + a)^{n+1} = (1 + a^{2^k})^m = 1 + a^{2^k} + \binom{m}{2} a^{2 \cdot 2^k} + \dots$$

It follows that $w(P^n)$ is trivial if and only if $2^k = n + 1$. □

**Applications.** A (not necessarily associative) algebra is called a *division algebra* if every equation of the form $ax = b$ and $xa = b$ with nonzero $a$ and arbitary $b$ has a unique solution.

**Lemma 20.** *A finite-dimensional algebra is a division algebra if and only if it has no zero divisors.*

*Proof.* Left and right multiplication are linear endomorphisms of a finite-dimensional vector space, hence injective if and only if surjective. □

**Theorem 21** (Stiefel)**.** *Suppose there exists a real division algebra of dimension $n$. Then the projective space $P^{n-1}$ is parallelizable and $n$ is a power of 2.*

*Proof.* Up to isomorphism, any real division algebra of dimension $n$ is of the form $(\mathbb{R}^n, +)$ with a bilinear product $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ without zero divisors. Let us denote the standard basis of $\mathbb{R}^n$ by $(e_i)$. Since $p$ has no zero divisors, it induces automorphisms $p(\cdot, e_i)$ and

$$v_i := p(\cdot, e_i) p(\cdot, e_1)^{-1}$$

Note that $v_1 = \mathrm{id}$ and $(v_i(x))$ are linearly independent for $x \neq 0$:

$$\sum \lambda_i v_i(x) = 0 \;\Rightarrow\; p(x, \sum \lambda_i e_i) = 0 \;\Rightarrow\; \lambda_i \equiv 0 \text{ or } x = 0$$

We can thus define sections of the bundle $\mathrm{Hom}(\gamma^1_{n-1}, \gamma^\perp_{n-1}) \cong \tau_{P^{n-1}}$ as follows:

$$s_i([x])(y) := \text{ orthogonal projection of } v_i(y) \text{ along } \langle x \rangle$$

Since $s_1 \equiv 0$ it follows that $s_2, \ldots, s_n$ are $n-1$ nowhere linearly dependent sections. We have thus proved that $P^{n-1}$ is parallelizable, and now Cor. 19 shows that $n$ must be a power of 2. □

In fact, one can show that the projective space $P^{n-1}$ is parallelizable only for $n = 1$, 2, 4 or 8. It follows that finite-dimensional real division algebras exist precisely in these dimensions!

## References

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