Hilbert-Schmidt and trace class operators*

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Let $H \neq 0$ be a Hilbert space. We denote by $B(H)$ and $K(H)$ the algebra of bounded respective compact operators on H and by $B_{\text{fin}}(H)$ the subspace of operator of finite rank. We write $\cong$ if two spaces are isometrically isomorphic. The space of bounded sequences with index set J is denoted by $l^\infty(J)$, its (closed) subspace of zero sequences by $c_0(J)$ and the subspace of sequences with finite support by $c_{\text{fin}}(J)$. The space of (square) summable sequences is written as $l^2(J)$ and $l^1(J)$, respectively.

1 Introduction

Recall that we have the following hierarchy classic sequence spaces:

$$c_{\text{fin}}(\mathbb{N}) \subseteq l^1(\mathbb{N}) \subseteq l^2(\mathbb{N}) \subseteq c_0(\mathbb{N}) \subseteq l^\infty(\mathbb{N})$$

They are Banach spaces (for $c_{\text{fin}}(\mathbb{N})$) and commutative algebras; $l^2(\mathbb{N})$ even is a Hilbert space.

Similarly, we have the following chain of **operator algebras**:

$$B_{\text{fin}}(H) \subseteq ? \subseteq ? \subseteq K(H) \subseteq B(H)$$

They are Banach spaces (except for $B_{\text{fin}}(H)$) and algebras, although **non-commutative** in general.

The following proposition shows that we can in a sense interprete these operator algebras as the non-commutative analoga of the respective sequence spaces.

1 Proposition. For any orthonormal system (e_n) in H we have an isometric algebra homomorphism

$$\Phi : l^\infty \to B(H), (a_n) \mapsto x \mapsto \sum_n a_n \langle x, e_n \rangle e_n$$

with $\Phi^{-1}(B_{\text{fin}}(H)) = c_{\text{fin}}$ and $\Phi^{-1}(K(H)) = c_0$.

Proof. We only prove that Φ is well-defined and an isometry:

$$||\sum_n a_n \langle x, e_n \rangle e_n||^2 = \sum_n |a_n|^2 |\langle x, e_n \rangle|^2$$
$$\leqslant ||(a_n)||_{l^2}^2 \sum_n |\langle x, e_n \rangle|^2 \leqslant ||(a_n)||_{l^2}^2 ||x||^2$$

and $||\sum_n a_n \langle e_m, e_n \rangle e_n|| = |a_m|$, hence $||\Phi((a_n))|| = ||(a_n)||_{l^2}$. □

It is thus natural to ask the following **questions**: (1) What operator algebras correspond to $l^1(\mathbb{N})$ and $l^2(\mathbb{N})$? (2) Which familiar results from the theory of sequence spaces generalize to the non-commutative case?

*based on [Nee96]

2 Hilbert-Schmidt operators

We define the space of **Hilbert-Schmidt operators** as

$$B_2(H) := \{A \in B(H) : ||A||_2 < \infty\}$$
$$||A||_2 := \sqrt{\sum_{i \in I} ||Ae_i||^2}$$

where $(e_i)_{i \in I}$ is an ONB of H. This is a normed space.

An easy calculation shows that this definition does not depend on the choice of basis:

2 Lemma. Let $A \in B(H)$ and let $(e_i)_{i \in I}$, $(f_j)_{j \in J}$ be two ONBs. Then:

$$\sum_{i \in I} ||Ae_i||^2 = \sum_{j \in J} ||A^* f_j||^2 \in [0, \infty]$$

Proof. Suppose the first limit exist. Then by Fubini we have

$$\sum_{i \in I} ||Ae_i||^2 = \sum_{i \in I} \sum_{j \in J} |\langle Ae_i, f_j \rangle|^2$$
$$= \sum_{j \in J} \sum_{i \in I} |\langle A^* f_j, e_i \rangle|^2 = \sum_{j \in J} ||A^* f_j||^2,$$

hence the second limit exists and agrees. □

The following facts follow easily from the preceding.

3 Proposition. Let $A \in B_2(H)$. Then:

(i) $||A^*||_2 = ||A||_2$

(ii) $||A|| \leqslant ||A||_2$

(iii) $B_2(H)$ is an "operator ideal" in $B(H)$, i.e. $B(H) B_2(H) B(H) \subseteq B_2(H)$

Proof. (i) Lemma 2.

(ii) Let $\epsilon > 0$. Take $e \in H$ such that $||e|| = 1$ and $||Ae|| >= ||A|| - \epsilon$, extend to an ONB (e_i). Then

$$||A||_2^2 = \sum_i ||Ae_i||^2 \geqslant ||Ae||^2 >= (||A|| - \epsilon)^2$$

(iii) It is clear that $B_2(H)$ is a left ideal; (i) shows that it is a right ideal. □

4 Theorem. $(B_2(H), ||\cdot||_2)$ is a Hilbert space with inner product

$$\langle A, B \rangle_2 := \sum_i \langle B^* A e_i, e_i \rangle = \sum_i \langle A e_i, B e_i \rangle$$

and $B_{\text{fin}}(H)$ is a dense subspace.

Proof. Consider the mapping

$$\Psi : c_{\text{fin}}(I \times I) \to B_2(H), \delta_{(i,j)} \mapsto \langle \cdot, e_i \rangle e_j$$

From the calculations

$$||\Psi((a_{i,j}))||^2 \leqslant ||\Psi((a_{i,j}))||_2^2 = ||\sum_{i,j} a_{i,j} \langle \cdot, e_i \rangle e_j||_2^2$$
$$= \sum_{i,j} |a_{i,j}|^2 = ||(a_{i,j})||_{l^2}^2$$

we see that Ψ has a continuous extension $l^2(I \times I) \to B(H)$ which is a surjective isometry onto $B_2(H)$. Thus the latter is also a Hilbert space with dense subspace $\Psi(c_{\text{fin}}(I \times I)) = B_{\text{fin}}(H)$.

The formula for the inner product is easily obtained using the polarization identity. ☐

5 Corollary. $B_2(H) \subseteq K(H)$

Proof. Theorem 4 and Proposition 3, (ii). ☐

Any Hilbert-Schmidt operator $A \in B_2(H) \subseteq K(H)$ can be written as a series

$$A = \sum_n a_n \langle \cdot, e_n \rangle f_n$$

with $(a_n) \in c_0(\mathbb{N})$ and orthonormal systems (e_n), (f_n). We can easily calculate its norm from any such representation:

6 Proposition.

$$||A||_2 = \sqrt{\sum_n |a_n|^2} = ||(a_n)||_{l^2}$$

Thus a compact operator is a Hilbert-Schmidt operator if and only if its coefficients are in $l^2(\mathbb{N})$.

Finally we will reveal the intimate connection between the Hilbert-Schmidt operators on H and the tensor product of H with its dual.

7 Proposition. The space of Hilbert-Schmidt operator is naturally isometrically isomorphic to the tensor product $H^* \otimes H$ via

$$\Phi : H^* \otimes H \to B_2(H), \lambda \otimes f \mapsto \lambda f$$

Proof. The mapping Φ is induced by the bilinear map $(\lambda, f) \mapsto \lambda f$, hence well-defined. Choose an ONB (e_i) of H. Then $\langle \cdot, e_i \rangle \otimes e_j$ is an ONB of the tensor product $H^* \otimes H$, and

$$||\Phi(\sum_{i,j} a_{i,j} \langle \cdot, e_i \rangle \otimes e_j)||_2^2 = ||\sum_{i,j} a_{i,j} \langle \cdot, e_i \rangle e_j||_2^2$$
$$= \sum_{i,j} |a_{i,j}|^2 = ||\sum_{i,j} a_{i,j} \langle \cdot, e_i \rangle \otimes e_j||^2$$

shows that Φ is an isometry. Thus Φ is also surjective since its range includes the dense set of finite-rank operators. Now apply the bounded inverse theorem. ☐

3 Trace class operators

We define the space of **trace class** (or **nuclear**) **operators** to be

$$B_1(H) := \{A \in B_2(H) : ||A||_1 < \infty\}$$
$$||A||_1 := \sup\{|\langle A, B \rangle_2| : B \in B_2(H), ||B|| \leqslant 1\}$$

This is a normed space.

Let us first collect some facts about this space.

8 Proposition. Let $A \in B_1(H)$. Then:

(i) $||A^*||_1 = ||A||_1$

(ii) $||A||_2 \leqslant ||A||_1$

(iii) $B_1(H)$ is an "operator ideal" in $B(H)$, i.e. $B(H)B_1(H)B(H) \subseteq B_1(H)$

(iv) $B_2(H)B_2(H) \subseteq B_1(H)$

Proof. (i) We have $\langle A, B \rangle_2 = \langle B^*, A^* \rangle_2$ since both sides define inner products inducing the same norm (apply the polarization identity). This in turn implies the claim.

(ii) This follows from $||A|| \leqslant ||A||_2$.

(iii) In view of (i) we only have to show that $B_1(H)$ is a left ideal; this follows readily from $\langle CA, B \rangle_2 = \langle A, C^* B \rangle_2$.

(iv) Let $A, B, C \in B_2(H)$ and $||B|| \leqslant 1$. Then

$$|\langle CA, B \rangle_2| = |\langle A, C^* B \rangle_2| \leqslant ||A||_2 ||C^* B||_2$$
$$= ||A||_2 ||B^* C||_2 \leqslant ||A||_2 ||B^*|| ||C||_2 \leqslant ||A||_2 ||C||_2,$$

hence $||CA||_1 \leqslant ||C||_2 ||A||_2$. ☐

We define the **trace** of a trace class operator $A \in B_1(H)$ to be

$$\text{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle$$

where $(e_i)_{i \in I}$ is an ONB of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional.

The following lemma shows that the definition make sense.

9 Lemma. The series converges absolutely and it is independent from the choice of basis.

Proof. Choose $\lambda_i \in \mathbb{C}$ such that $|\langle Ae_i, e_i \rangle| = \lambda_i \langle Ae_i, e_i \rangle$ and $|\lambda_i| = 1$ $(i \in I)$. Then for every finite subset $I_0 \subseteq I$ we have the following estimate:

$$\sum_i |\langle Ae_i, e_i \rangle| = \sum_i \lambda_i \langle Ae_i, e_i \rangle = \sum_i \lambda_i \langle A, \langle \cdot, e_i \rangle e_i \rangle_2$$
$$= \langle A, \sum_i \lambda_i \langle \cdot, e_i \rangle e_i \rangle_2 \leqslant ||A||_1 || \sum_i \lambda_i \langle \cdot, e_i \rangle e_i || \leqslant ||A||_1$$

This implies absolute convergence.

If $(f_j)_{j\in J}$ is any other ONB we have

$$\sum_i \langle Ae_i, e_i\rangle = \sum_{i,j}\langle Ae_i, f_i\rangle\langle f_i, e_i\rangle = \sum_{j,i}\langle f_i, e_i\rangle\langle e_i, A^* f_i\rangle$$
$$= \sum_j \langle f_i, A^* f_i\rangle = \sum_j \langle Af_i, f_i\rangle,$$

hence the trace is independent from the particular choice of basis. □

We now collect some facts about the trace which resemble the finite-dimensional case.

10 Proposition. (i) $\operatorname{tr} \in B_1(H)'$ with $||\operatorname{tr}|| = 1$

(ii) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for $A \in B_1(H)$, $B \in B(H)$ and $A, B \in B_2(H)$, respectively

Proof. (i) By the proof of the preceding lemma we have $||\operatorname{tr}|| \leqslant 1$. Equality follows by considering an orthogonal projection.

(ii) If $A \in B_1(H)$ is Hermitian and $B \in B(H)$ we can take an ONB of eigenvectors (e_i) with $Ae_i =: \lambda_i e_i$ for real eigenvalues $\lambda_i \in \mathbb{R}$. Then

$$\operatorname{tr}(AB) = \sum_i \langle ABe_i, e_i\rangle = \sum_i \langle Be_i, Ae_i\rangle$$
$$= \sum_i \langle BAe_i, e_i\rangle = \operatorname{tr}(BA)$$

If $A \in B_1(H)$ is a general trace class operator we can still write it as a sum $A = B + iC$ with Hermitian B, $C \in B_1(H)$. The claim then follows from the complex bilinearity of $(A, B) \mapsto \operatorname{tr}(AB)$ and $(A, B) \mapsto \operatorname{tr}(BA)$.

For $A, B \in B_2(H)$ the claim follows from

$$\operatorname{tr}(AB) = \sum_i \langle ABe_i, e_i\rangle = \langle B, A^*\rangle$$
$$= \langle A, B^*\rangle = \sum_i \langle BAe_i, e_i\rangle = \operatorname{tr}(BA)$$ □

Let us write a trace class operator $A \in B_1(H)$ as a series

$$A = \sum_n a_n \langle \cdot, e_n\rangle f_n$$

with $(a_n) \in c_0(\mathbb{N})$ and orthonormal systems (e_n), (f_n). Again it is easy to calculate its norm and trace from this representation:

11 Proposition.

$$||A||_1 = \sum_n |a_n| = ||(a_n)||_{l^1}$$
$$\operatorname{tr}(A) = \sum_n a_n \langle f_n e_n\rangle$$

Thus a compact operator is a trace class operator if and only if its coefficients are in $l^1(\mathbb{N})$.

Proof. We only show the first equality; the second one is immediate from the definition of tr.

($\leqslant$) For any $B \in B_2(H)$ with $||B|| \leqslant 1$ we have

$$|\langle A, B\rangle_2| \leqslant \sum_n |a_n||\langle\langle \cdot, e_n\rangle f_n, B\rangle_2|$$
$$\leqslant \sum_n |a_n||\langle f_n, Be_n\rangle| \leqslant \sum_n |a_n|,$$

hence $||A||_1 \leqslant \sum_n |a_n|$.

($\geqslant$) Choose $b_n \in \mathbb{C}$ such that $|a_n| = a_n b_n$ and $|b_n| = 1$ ($n \in \mathbb{N}$) and define

$$B_N := \sum_{n=1}^{N} b_n \langle \cdot, e_n\rangle f_n$$

Clearly $B_N \in B_2(H)$ and $||B_N|| \leqslant 1$. Hence

$$||A||_1 \geqslant |\langle A, B_N\rangle_2| = |\sum_n \langle Ae_n, B_N e_n\rangle|$$
$$= |\sum_{n=1}^{N} a_n b_n| = \sum_{n=1}^{N} |a_n| \uparrow \sum_n |a_n|$$ □

It follows that we can approximate any trace class operator using finite rank operators:

12 Corollary. B_{fin} is a dense subspace of $(B_1(H), ||\cdot||_1)$.

Proof. We have

$$||A - \sum_{n=1}^{N} a_n \langle \cdot, e_n\rangle f_n||_1 \overset{11}{=} \sum_{n=N}^{\infty} |a_n| \to 0$$

as $N \to \infty$. □

We can also deduce that every trace class operator is the product of two Hilbert-Schmidt operators:

13 Proposition. $B_2(H)B_2(H) = B_1(H)$

Proof. ($\subseteq$) was proved in Proposition 8, (iv).

($\supseteq$) Define

$$B = \sum_n \sqrt{a_n}\langle \cdot, e_n\rangle f_n$$
$$C = \sum_n \sqrt{a_n}\langle \cdot, e_n\rangle e_n$$

Then B and C are Hilbert-Schmidt operators, and $A = BC$. □

Note that

$$B_1(H) \times B(H) \to \mathbb{C},\ (A, B) \mapsto \operatorname{tr}(AB)$$

is a continuous pairing since we have

$$|\operatorname{tr}(AB)| \leqslant ||AB||_1 \leqslant ||A||_1 ||B||$$

This pairing induces the following two isometric isomorphisms.

14 Theorem. $B_1(H) \cong K(H)'$ and $B_1(H)' \cong B(H)$

Proof. (1) We show that

$$B_1(H) \to K(H)', B \mapsto \mathrm{tr}(\cdot B)$$

is an isometric isomorphism.

Linearity is obvious. It is almost by definition of $||\cdot||_1$ that the mapping is an isometry. Hence it remains to show surjectivity. Let $\varphi \in K(H)'$. Then for all $A \in B_2(H)$ we have

$$|\varphi(A)| \leqslant ||\varphi|| ||A|| \leqslant ||\varphi|| ||A||_2,$$

hence $\varphi|_{B_2(H)} \in B_2(H)'$. Take the unique $B \in B_2(H)$ such that

$$\varphi|_{B_2(H)} = \langle \cdot, B\rangle_2 = \langle B^*, \cdot^*\rangle_2 = \mathrm{tr}(\cdot B^*)$$

From this we see that the continuity of φ implies that B^* is of trace class, and density of $B_2(H) \subseteq K(H)$ shows that B^* is a preimage of φ.

(2) We show that

$$B(H) \to B_1(H)', B \mapsto \mathrm{tr}(\cdot B)$$

is an isometric isomorphism.

Again, it is obvious that the mapping is a linear isometry. We show surjectivity. Let $\varphi \in B_1(H)'$. Since for $e, f \in H$

$$||\langle \cdot, e\rangle f||_1 = ||e|| ||f||$$

we see that the mapping $f \mapsto \varphi(\langle \cdot, e\rangle f)$ is in H'. Hence there is a unique $\varphi_e \in H$ such that

$$\langle f, \varphi_e\rangle = \varphi(\langle \cdot, e\rangle f) \qquad \forall f \in H$$

and $||\varphi_e|| \leqslant ||\varphi|| ||e||$. Thus $B : H \to H, e \mapsto \varphi_e$ defines a bounded operator. And the calculation

$$\varphi(\langle \cdot, e\rangle f) = \langle f, Be\rangle = \langle \langle \cdot, e\rangle f, B\rangle_2 = \mathrm{tr}(\langle \cdot, e\rangle f B^*)$$

together with density of $B_{\mathrm{fin}}(H) \subseteq B_1(H)$ shows that B^* is a preimage of φ. □

15 Corollary. $(B_1(H), ||\cdot||_1)$ is a Banach space.

4 Summary

Propositions 6 and 11 show that the algebras of Hilbert-Schmidt and trace class operators are the natural non-commutative analoga of $l^1(\mathbb{N})$ and $l^2(\mathbb{N})$, respectively. That is, we have the following chains which correspond in the sense of Proposition 1:

$$c_{\mathrm{fin}}(\mathbb{N}) \subseteq l^1(\mathbb{N}) \subseteq l^2(\mathbb{N}) \subseteq c_0(\mathbb{N}) \subseteq l^\infty(\mathbb{N})$$
$$B_{\mathrm{fin}}(H) \subseteq B_1(H) \subseteq B_2(H) \subseteq K(H) \subseteq B(H)$$

In table 1 we have summarized some familiar facts about sequence spaces together with their non-commutative counterparts (which we have proved in the preceding).

Table 1: Comparison of sequence and operator spaces

Sequence spaces	Operator spaces
$c_{\mathrm{fin}}(\mathbb{N})$ dense in $l^1(\mathbb{N})$, $l^2(\mathbb{N})$ and $c_0(\mathbb{N})$	$B_{\mathrm{fin}}(H)$ dense in $B_1(H)$, $B_2(H)$ and $K(H)$
$l^1(\mathbb{N}) = l^2(\mathbb{N})l^2(\mathbb{N})$	$B_1(H) = B_2(H)B_2(H)$
$(a_n) \mapsto \sum_n a_n \in l^1(\mathbb{N})'$	$\mathrm{tr} \in B_1(H)'$
$c_0(\mathbb{N})' \cong l_1(\mathbb{N})$	$K(H)' \cong B_1(H)$
$l_1(\mathbb{N})' \cong l^\infty(\mathbb{N})$	$B_1(H)' \cong B(H)$

References

[Nee96] Karl-Hermann Neeb. Skript zur Vorlesung Spektral- und Darstellungstheorie. http://www.mathematik.tu-darmstadt.de/fbereiche/AlgGeoFA/staff/neeb/skripten/Spektraltheorie-SS96.pdf, 1996.