Helmholtz projection and Stokes operator on arbitrary domains

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Let $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ denote the space of test functions on Ω, and let $\mathcal{D}'(\Omega)$ be its dual, the space of distributions. All function spaces are real.

1 Introduction

For sufficiently "nice" [1]domains $\Omega \subseteq \mathbb{R}^n$, one has the *Helmholtz decomposition*

$$L^p(\Omega)^n = L^p_\sigma(\Omega) \oplus G^p(\Omega)$$

of L^p into a sum of the *solenoidal* subspace L^p_σ and the space of gradients G^p; that is:

$$\mathcal{D}_\sigma(\Omega) = \{u \in \mathcal{D}(\Omega)^n \mid \operatorname{div}(u) = 0\}$$
$$L^p_\sigma(\Omega) = \overline{\mathcal{D}_\sigma(\Omega)}^{||\cdot||_{L^p}}$$
$$G^p(\Omega) = \{\nabla p \mid p \in L^p_{loc}(\Omega) \text{ and } \nabla p \in L^p(\Omega)^n\}$$

The projection $\mathbb{P} : L^p(\Omega)^n \to L^p_\sigma(\Omega)$ is called *Helmholtz projection*.

It is commonly applied[2]to the Navier-Stokes equation

$$\partial_t u - \Delta u + \nabla \pi + (u \cdot \nabla) u = 0$$

in order to eliminate the pressure term (which is a gradient). In doing so, one introduces new terms such as $-\mathbb{P}\Delta u$, which can as well be understood to be the application of an operator $-\mathbb{P}\Delta$, the so-called *Stokes operator*.

In this article we will first describe MONNIAUX's ([2]) approach of generalizing both Helmholtz projection and Stokes operator to arbitrary domains in the special case $n = 3$ and $p = 2$.

In the final section, we will briefly consider the general case and show a characterization of the Helmholtz projection by HAAK and KUNSTMANN [1] (based on the approach of SIMADER and SOHR in [4]).

Before getting started, we cite the following

1 Fact. *(i) (de Rahm) Let $u \in \mathcal{D}'(\Omega)^n$. Then there is an $p \in \mathcal{D}'(\Omega)$ with $u = \nabla p$ if and only if u vanishes on $\mathcal{D}_\sigma(\Omega)$.*

(ii) Let $p \in \mathcal{D}'(\Omega)$. If $\nabla p \in L^q(\Omega)^n$, then $p \in L^q_{loc}(\Omega)$.

2 Helmholtz projection

Let $\Omega \subseteq \mathbb{R}^3$ be an *arbitrary* domain. The space

$$L^2(\Omega)^3 \quad \text{with} \quad \langle u, v\rangle = \int_\Omega u \cdot v \, d\mu$$

is a Hilbert space, and if we define

$$G^2(\Omega) := \{\nabla p \mid p \in L^2_{loc}(\Omega) \text{ and } \nabla p \in L^2(\Omega)^3\}$$
$$L^2_\sigma(\Omega) := G^2(\Omega)^\perp$$

[1]e.g. for bounded, simply connected domains with Lipschitz or C^2-boundary.

[2]Of course this is only possible if u satisfies certain properties, e.g. $-\Delta u \in L^p(\Omega)$.

we have the *Helmholtz decomposition*

$$L^2(\Omega)^3 = L^2_\sigma(\Omega) \overset{\perp}{\oplus} G^2(\Omega)$$

since $G^2(\Omega)$ is a closed subspace of $L^2(\Omega)^3$ (which follows from fact 1 (i)).

The following proposition shows that this is a Helmholtz decomposition in the sense of the introduction.

2 Proposition.

$$L^2_\sigma(\Omega) = \overline{\mathcal{D}_\sigma(\Omega)}^{||\cdot||_{L^2}}$$

Proof. (1) Let $\phi \in \mathcal{D}_\sigma(\Omega)$. For all $\nabla p \in G^2(\Omega)$ we have

$$\langle \phi, \nabla p \rangle = - \langle \operatorname{div}(\phi), p \rangle = 0$$

(by partial integration), hence $\phi \in L^2_\sigma(\Omega)$.

(2) In order to prove density it is sufficient to show that the following statement holds for all $u \in L^2(\Omega)^3$:

$$u \perp \mathcal{D}_\sigma(\Omega) \quad \Rightarrow \quad u \in G^2(\Omega)$$

But this is clear from fact 1. □

The orthogonal projection $\mathbb{P} : L^2(\Omega)^3 \to L^2_\sigma(\Omega)$ is again called *Helmholtz projection*. It can also be characterized as follows:

3 Proposition. *The Helmholtz projection $\mathbb{P}$ is the adjoint of the canonical injection $J : L^2_\sigma(\Omega) \to L^2(\Omega)^3$, that is, $\mathbb{P} = J^*$.*

Proof. Let $u \in L^2_\sigma(\Omega)$ and $v \in L^2(\Omega)^3$ with Helmholtz decomposition $v = v_\sigma + v_\nabla$. Evidently,

$$\langle u, \mathbb{P}v \rangle = \langle u, v_\sigma \rangle = \langle u, v \rangle = \langle Ju, v \rangle$$

□

3 Dirichlet-Laplace operator

There are various definitions of the Laplace operator Δ (e.g. the classical definition, weak definitions, the distributional definition etc.). It will turn out that for our purposes it is useful to employ the *Dirichlet-Laplace operator* whose construction we will give in what follows.

We consider the Hilbert space

$$H^1_0(\Omega)^3 = \overline{\mathcal{D}(\Omega)^3}^{\langle \cdot,\cdot \rangle_{H^1}} \subseteq H^1(\Omega)^3$$

$$\text{with} \quad \langle u, v \rangle_{H^1} = \langle u, v \rangle + \sum_{i=1}^{3} \langle \partial_i u, \partial_i v \rangle$$

whose dual is commonly designated by

$$H^{-1}(\Omega)^3 := (H^1_0(\Omega)^3)'$$

Now the bilinear form

$$\mathfrak{a} : \begin{cases} H^1_0(\Omega)^3 \times H^1_0(\Omega)^3 \to \mathbb{R} \\ (u, v) \mapsto \sum_{i=1}^{3} \langle \partial_i u, \partial_i v \rangle \end{cases}$$

induces the *Dirichlet-Laplace*[3] operator given by

$$\Delta^\Omega_D : \begin{cases} H^1_0(\Omega)^3 \to H^{-1}(\Omega)^3 \\ v \mapsto -\mathfrak{a}(\cdot, v) \end{cases}$$

4 Proposition. *Δ^Ω_D is well-defined and bounded.*

[3] This name reflects the fact that Δ^Ω_D is defined on the function space $H^1_0(\Omega)$ consisting of functions u satisfying the Dirichlet boundary condition $u|_{\delta\Omega} = 0$.

Proof.

$$\begin{aligned} |-\mathfrak{a}(u,v)| &\le \sum_{i=1}^{3} |\langle \partial_i u, \partial_i v\rangle| \;\le\; \sum_{i=1}^{3} ||\partial_i u|| ||\partial_i v|| \\ &\le \left(\sum_{i=1}^{3} ||\partial_i u||^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{3} ||\partial_i v||^2\right)^{\frac{1}{2}} \\ &\le ||u||_{H^1} ||v||_{H^1} \end{aligned}$$

$$\Rightarrow ||-\mathfrak{a}(\cdot,v)|| \le ||v||_{H^1} \quad \text{and} \quad ||\Delta_D^{\Omega}|| \le 1$$

□

We remark that the operator

$$\delta - \Delta_D^{\Omega} : \begin{cases} H_0^1(\Omega)^3 \to H^{-1}(\Omega)^3 \\ v \mapsto \delta \langle \cdot, v\rangle + \mathfrak{a}(\cdot, v) \end{cases}$$

is an isomorphism for all $\delta > 0$. If the domain Ω is bounded, Poincaré's inequality shows that even Δ_D^{Ω} is an isomorphism.

4 Stokes operator

Now that we have a useful definition of the Laplace operator we need to "extend" the Helmholtz projection such that its domain contains $H^{-1}(\Omega)$.

The space

$$\mathcal{V} := H_0^1(\Omega)^3 \cap L_\sigma^2(\Omega)$$

is a closed subspace of $H_0^1(\Omega)^3$, and hence a Hilbert space in its own right (endowed with $\langle \cdot, \cdot\rangle_{H^1}$).

5 Proposition.

$$L_\sigma^2(\Omega) = \overline{\mathcal{V}}^{||\cdot||_{L^2}}$$

Proof. Since $\mathcal{D}_\sigma(\Omega) \subseteq \mathcal{V}$ this follows from proposition 2. □

This already indicates that $\mathcal{V}$ is still large enough a space to be useful It will also prove useful later on.

Let $\tilde{J} : \mathcal{V} \to H_0^1(\Omega)^3$ be the canonical injection and let $\tilde{\mathbb{P}} : H^{-1}(\Omega) \to \mathcal{V}'$ be its adjoint. Clearly both are continuous mappings.

6 Proposition. *$\tilde{\mathbb{P}}$ and $\mathbb{P}$ are consistent in the following sense*

$$\tilde{\mathbb{P}}(\langle \cdot, v\rangle) = \langle \cdot, \mathbb{P}v\rangle \quad \forall v \in L^2(\Omega)^3$$

Proof. For all $u \in \mathcal{V}$ we have

$$\tilde{\mathbb{P}}(\langle \cdot, v\rangle)(u) = \left\langle \tilde{J}u, v \right\rangle = \langle Ju, v\rangle = \langle u, \mathbb{P}v\rangle$$

□

7 Proposition. *The operator*

$$A_0 : \begin{cases} \mathcal{V} \to \mathcal{V}' \\ v \mapsto -\tilde{\mathbb{P}}\Delta_D^{\Omega}\tilde{J}v \end{cases}$$

is bounded and self-adjoint in $(\mathcal{V}, \langle \cdot, \cdot\rangle_{H^1})$, and $A_0 v = \mathfrak{a}(\cdot, v)$ where $\mathfrak{a}$ is the bilinear form from section 3.

Proof. (1) A_0 is composed of bounded operators, hence bounded.

(2) For all $u, v \in \mathcal{V}$ we have

$$\begin{aligned} (A_0 v)(u) &= (-\tilde{\mathbb{P}}\Delta_D^{\Omega}\tilde{J}v)(u) = -(\Delta_D^{\Omega}\tilde{J}v)(\tilde{J}u) \\ &= \mathfrak{a}(\tilde{J}u, \tilde{J}v) = \mathfrak{a}(u, v) \end{aligned}$$

(3) A_0 is defined on the entire Hilbert space $\mathcal{V}$ (i.e. bounded), hence self-adjointness will follow from symmetry. But it is clear from (2) that A_0 is symmetric. □

8 Definition. The *Stokes operator* is defined by

$$\begin{aligned} D(A) :=& \{v \in \mathcal{V} \mid A_0 v \in L^2_\sigma(\Omega)\} \\ :=& \{v \in \mathcal{V} \mid \exists u \in L^2_\sigma(\Omega) : A_0 v = \langle \cdot, u\rangle\} \\ A : D(A) \to L^2_\sigma(\Omega),& \quad v \mapsto u \end{aligned}$$

9 Theorem. *The Stokes operator is self-adjoint in* $(L^2_\sigma(\Omega), \langle\cdot,\cdot\rangle)$*, generates an analytic semigroup* $\left(e^{-tA}\right)_{t\geq 0}$*, and satisfies*

$$\begin{aligned} D(A) =& \{v \in \mathcal{V} \mid \exists \pi \in \mathcal{D}'(\Omega) : \nabla\pi \in H^{-1}(\Omega)^3 \\ & \text{and } -\Delta^\Omega_D v + \nabla\pi \in L^2_\sigma(\Omega)\} \\ :=& \{v \in \mathcal{V} \mid \exists \pi \in \mathcal{D}'(\Omega) : \nabla\pi \in H^{-1}(\Omega)^3 \\ & \text{and } \exists u \in L^2_\sigma(\Omega) : -\Delta^\Omega_D v + \nabla\pi = \langle\cdot, u\rangle\} \\ Av =& u \end{aligned}$$

Proof. (1) By proposition 5 it follows that $\mathfrak{b} := \mathfrak{a}|_{\mathcal{V}\times\mathcal{V}}$ is a densely-defined bilinear form on $L^2_\sigma(\Omega)$. It is also:

(i) accretive, as $\mathfrak{b} \geq 0$,

(ii) closed, since $D(\mathfrak{b}) = \mathcal{V}$, $||\cdot||_{\mathfrak{b}} = ||\cdot||_{H^1}$ and we have seen before that $(\mathcal{V}, \langle\cdot,\cdot\rangle_{H^1})$ is complete,

(iii) continous in the sense that $|\mathfrak{b}(u,v)| \leq C||u||_{\mathfrak{b}}||v||_{\mathfrak{b}}$; this is shown by proposition 4.

Under these conditions, the operator associated with the adjoint form $\mathfrak{b}^*$ is simply the adjoint of the operator associated with $\mathfrak{b}$ (see e.g. OUHABAZ [3]).

A is *by its very definition* the operator associated with $\mathfrak{b}$. And $\mathfrak{b}$ is symmetric, hence $\mathfrak{b}^* = \mathfrak{b}$ and it follows that $A^* = A$, $D(A^*) = D(A)$, that is, A is self-adjoint.

(2) Another result from the theory of sesquilinear forms is that if A is the operator associated with a densely-defined bilinear form $\mathfrak{b}$ satisfying (i)-(iii), $-A$ generates a (strongly continuous contraction) semigroup on $L^2_\sigma(\Omega)$ which is holomorphic in an open sector

$$\{z \in \mathbb{C} \mid 0 \neq z \text{ and } |\arg(z)| < \theta\}$$

In particular it is analytic on $\mathbb{R}_{>0}$.

(3) Let $Av = u$. Then

$$\begin{aligned} & -\tilde{\mathbb{P}}\Delta^\Omega_D v = \langle\cdot, u\rangle \overset{6}{=} \tilde{\mathbb{P}}(\langle\cdot,u\rangle) \\ \Rightarrow\ & \tilde{\mathbb{P}}\left(-\Delta^\Omega_D v - \langle\cdot,u\rangle\right) = 0 \\ \Rightarrow\ & -\Delta^\Omega_D v - \langle\cdot,u\rangle \in \ker(\tilde{\mathbb{P}}) \end{aligned}$$

It remains to show that the kernel consists of gradients:

$$\begin{aligned} \ker(\tilde{\mathbb{P}}) =& \{\phi \in H^{-1}(\Omega)^3 \mid \tilde{\mathbb{P}}\phi = 0\} \\ =& \{\phi \in H^{-1}(\Omega)^3 \mid \phi|_{\mathcal{V}} = 0\} \\ \overset{2}{=}& \{\phi \in H^{-1}(\Omega)^3 \mid \phi|_{\mathcal{D}_\sigma(\Omega)} = 0\} \\ \overset{1}{=}& \{\phi \in H^{-1}(\Omega)^3 \mid \exists \pi \in \mathcal{D}'(\Omega) : \phi = \nabla\pi\} \end{aligned}$$

□

5 General case

Let $p \in (1,\infty)$, $1 = \frac{1}{p} + \frac{1}{p'}$, $n \geq 2$ and $\Omega \subseteq \mathbb{R}^n$ be an arbitrary domain. Let $\langle u, v\rangle := \int_\Omega u \cdot v \, d\mu$. Define

$$\begin{aligned} \dot{W}^{1,p}(\Omega) :=& \{[u] = u + \mathbb{C} \mid u \in L^p_{loc}(\Omega) \\ & \text{and } \nabla u \in L^p(\Omega)^n\} \\ ||[u]||_{\dot{W}^{1,p}} :=& ||\nabla u||_{L^p} \end{aligned}$$

This is a Banach space, as is its dual $\dot{W}^{1,p}(\Omega)'$. The gradient operator

$$\nabla_p : \dot{W}^{1,p}(\Omega) \to L^p(\Omega)^n, \quad u \mapsto \nabla u$$

is an isometry pretty much by definition of the norms. We isometrically identify $L^{p'}(\Omega)$ and $(L^p(\Omega))'$, and it follows that the adjoint

$$\nabla_p' : L^{p'}(\Omega)^n \to (\dot{W}^{1,p}(\Omega))'$$

is surjective with norm ≤ 1. Define

$$\begin{aligned} G^p(\Omega) &:= \operatorname{ran}(\nabla_p) \subseteq L^p(\Omega)^n \\ L^p_\sigma(\Omega) &:= \ker(\nabla_{p'}') \subseteq L^p(\Omega)^n \end{aligned}$$

Clearly this definition of G^p agrees with the one in the introduction. The following proposition shows that this also applies to L^p_σ.

10 Proposition.

$$L^p_\sigma(\Omega) = \overline{\mathcal{D}_\sigma(\Omega)}^{||\cdot||_{L^p}}$$

Proof. (1) Let $\phi \in \mathcal{D}_\sigma(\Omega)$. Then for all $u \in \dot{W}^{1,p'}(\Omega)$, we have

$$\begin{aligned} &(\nabla_{p'}'\phi)(u) = \langle \nabla_{p'} u, \phi \rangle = -\langle u, \operatorname{div}(\phi) \rangle = 0 \\ \Rightarrow &\phi \in \ker(\nabla_{p'}') = L^p_\sigma \end{aligned}$$

(2) In order to prove density it is sufficient to show that the values of a functional on $L^p_\sigma(\Omega)$ is already determined by its values on $\mathcal{D}_\sigma(\Omega)$. Equivalently, we show that the following assertion holds for all $v \in L^{p'}(\Omega)^n \cong (L^p(\Omega)^n)'$:

$$\langle \cdot, v \rangle |_{\mathcal{D}_\sigma(\Omega)} = 0 \quad \Rightarrow \quad \langle \cdot, v \rangle |_{L^p_\sigma(\Omega)} = 0$$

By the assumption and fact 1, we have $v = \nabla_{p'} p$ for a $p \in L^{p'}(\Omega)$, hence

$$\forall u \in L^p_\sigma(\Omega) : \quad \langle u, v \rangle = \langle u, \nabla_{p'} p \rangle = \left\langle \nabla_{p'}' u, p \right\rangle = 0$$

□

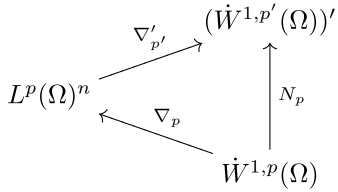
11 Theorem. *We have the decomposition*

$$L^p(\Omega)^n = L^p_\sigma(\Omega) \oplus G^p(\Omega)$$

if and only if the operator

$$N_p : \dot{W}^{1,p}(\Omega) \to (\dot{W}^{1,p'}(\Omega))', \quad N_p := \nabla_{p'}' \nabla_p$$

is bijective.



In that case,

$$\mathbb{P}_p : L^p(\Omega)^n \to L^p_\sigma(\Omega), \quad \mathbb{P}_p := \operatorname{id} - \nabla_p N_p^{-1} \nabla_{p'}'$$

is a projection with kernel $G^p(\Omega)$. It is called the Helmholtz projection *in $L^p(\Omega)$.*

Proof. ($\Rightarrow$) Since the sum is direct, we have

$$\ker(\nabla_{p'}') \cap \operatorname{ran}(\nabla_p) = \{0\}$$

and N_p is injective. Also,

$$\begin{aligned} &\ker(\nabla_{p'}') + \operatorname{ran}(\nabla_p) = L^p(\Omega)^n \\ \Rightarrow &\operatorname{ran}(N_p) = \nabla_{p'}'(\operatorname{ran}(\nabla_p)) = \nabla_{p'}'(L^p(\Omega)^n) = \operatorname{ran}(\nabla_{p'}') \end{aligned}$$

and N_p is surjective since $\nabla_{p'}'$ is so.

($\Leftarrow$) If N_p is bijective, its inverse is bounded by the open mapping theorem, hence $\mathbb{P}_p$ is bounded. Clearly,

$$\mathbb{P}_p|_{L^p_\sigma(\Omega)} = \operatorname{id}$$

and

$$\mathbb{P}_p v = 0 \;\Rightarrow\; v \in \operatorname{ran}(\nabla_p) = G^p(\Omega)$$
$$v = \nabla_p u \in G^p(\Omega) \;\Rightarrow\; \mathbb{P}_p v = v - \underbrace{\nabla_p N_p^- 1 N_p u}_{=\nabla_p u = v} = 0$$
$$\Rightarrow \ker(\mathbb{P}_p) = G^p(\Omega)$$

Hence, the sum $L^p_\sigma(\Omega) + G^p(\Omega)$ is direct, and and since every $v \in L^p(\Omega)^n$ can be written as

$$v = \underbrace{\mathbb{P}_p v}_{\in L^p_\sigma(\Omega)} + \underbrace{(v - \mathbb{P}_p v)}_{\in \ker(\mathbb{P}_p) = G^p(\Omega)}$$

it is entire space. □

Note that for $\phi \in \mathcal{D}(\Omega)$, $v \in \dot{W}^{1,p'}(\Omega)$ we have

$$(N_p \phi)(v) = \langle \nabla_p \phi, \nabla_{p'} v \rangle = \langle -\Delta \phi, v \rangle$$

hence $-N_p$ can be interpreted as a weak formulation of the Laplace operator.

From chapter 2, we see that for $n = 3$, N_2 is always bijective and $\mathbb{P}_2$ is just the orthogonal projection $\mathbb{P}$ onto $L^2_\sigma(\Omega)$.

Finally we remark that $\mathbb{P}_p$ can be extended in a similar way as $\mathbb{P}$ was extended to $\tilde{\mathbb{P}}$ in chapter 4, see HAAK and KUNSTMANN [1] for details.

References

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