

Helmholtz projection and Stokes operator on arbitrary domains

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Let $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ denote the space of test functions on Ω , and let $\mathcal{D}'(\Omega)$ be its dual, the space of distributions. All function spaces are real.

1 Introduction

For sufficiently “nice”¹ domains $\Omega \subseteq \mathbb{R}^n$, one has the *Helmholtz decomposition*

$$L^p(\Omega)^n = L_\sigma^p(\Omega) \oplus G^p(\Omega)$$

of L^p into a sum of the *solenoidal* subspace L_σ^p and the space of gradients G^p ; that is:

$$\mathcal{D}_\sigma(\Omega) = \{u \in \mathcal{D}(\Omega)^n \mid \operatorname{div}(u) = 0\}$$

$$L_\sigma^p(\Omega) = \overline{\mathcal{D}_\sigma(\Omega)}^{\|\cdot\|_{L^p}}$$

$$G^p(\Omega) = \{\nabla p \mid p \in L_{loc}^p(\Omega) \text{ and } \nabla p \in L^p(\Omega)^n\}$$

The projection $\mathbb{P} : L^p(\Omega)^n \rightarrow L_\sigma^p(\Omega)$ is called *Helmholtz projection*.

It is commonly applied² to the Navier-Stokes equation

$$\partial_t u - \Delta u + \nabla \pi + (u \cdot \nabla)u = 0$$

in order to eliminate the pressure term (which is a gradient). In doing so, one introduces new terms such as $-\mathbb{P}\Delta u$, which can as well be understood to be the application of an operator $-\mathbb{P}\Delta$, the so-called *Stokes operator*.

In this article we will first describe MONNIAUX’s ([2]) approach of generalizing both Helmholtz projection and Stokes operator to arbitrary domains in the special case $n = 3$ and $p = 2$.

In the final section, we will briefly consider the general case and show a characterization of the Helmholtz projection by HAAK and KUNSTMANN [1] (based on the approach of SIMADER and SOHR in [4]).

Before getting started, we cite the following

1 Fact. (i) (*de Rahm*) Let $u \in \mathcal{D}'(\Omega)^n$. Then there is an $p \in \mathcal{D}'(\Omega)$ with $u = \nabla p$ if and only if u vanishes on $\mathcal{D}_\sigma(\Omega)$.

(ii) Let $p \in \mathcal{D}'(\Omega)$. If $\nabla p \in L^q(\Omega)^n$, then $p \in L_{loc}^q(\Omega)$.

2 Helmholtz projection

Let $\Omega \subseteq \mathbb{R}^3$ be an *arbitrary* domain. The space

$$L^2(\Omega)^3 \quad \text{with} \quad \langle u, v \rangle = \int_\Omega u \cdot v \, d\mu$$

¹e.g. for bounded, simply connected domains with Lipschitz or C^2 -boundary.

²Of course this is only possible if u satisfies certain properties, e.g. $-\Delta u \in L^p(\Omega)$.

is a Hilbert space, and if we define

$$G^2(\Omega) := \{\nabla p \mid p \in L_{loc}^2(\Omega) \text{ and } \nabla p \in L^2(\Omega)^3\}$$

$$L_\sigma^2(\Omega) := G^2(\Omega)^\perp$$

we have the *Helmholtz decomposition*

$$L^2(\Omega)^3 = L_\sigma^2(\Omega) \oplus G^2(\Omega)$$

since $G^2(\Omega)$ is a closed subspace of $L^2(\Omega)^3$ (which follows from fact 1 (i)).

The following proposition shows that this is a Helmholtz decomposition in the sense of the introduction.

2 Proposition.

$$L_\sigma^2(\Omega) = \overline{\mathcal{D}_\sigma(\Omega)}^{\|\cdot\|_{L^2}}$$

Proof. (1) Let $\phi \in \mathcal{D}_\sigma(\Omega)$. For all $\nabla p \in G^2(\Omega)$ we have

$$\langle \phi, \nabla p \rangle = -\langle \operatorname{div}(\phi), p \rangle = 0$$

(by partial integration), hence $\phi \in L_\sigma^2(\Omega)$.

(2) In order to prove density it is sufficient to show that the following statement holds for all $u \in L^2(\Omega)^3$:

$$u \perp \mathcal{D}_\sigma(\Omega) \quad \Rightarrow \quad u \in G^2(\Omega)$$

But this is clear from fact 1. □

The orthogonal projection $\mathbb{P} : L^2(\Omega)^3 \rightarrow L_\sigma^2(\Omega)$ is again called *Helmholtz projection*. It can also be characterized as follows:

3 Proposition. *The Helmholtz projection \mathbb{P} is the adjoint of the canonical injection $J : L_\sigma^2(\Omega) \rightarrow L^2(\Omega)^3$, that is, $\mathbb{P} = J^*$.*

Proof. Let $u \in L_\sigma^2(\Omega)$ and $v \in L^2(\Omega)^3$ with Helmholtz decomposition $v = v_\sigma + v_\nabla$. Evidently,

$$\langle u, \mathbb{P}v \rangle = \langle u, v_\sigma \rangle = \langle u, v \rangle = \langle Ju, v \rangle$$

□

3 Dirichlet-Laplace operator

There are various definitions of the Laplace operator Δ (e.g. the classical definition, weak definitions, the distributional definition etc.). It will turn out that for our purposes it is useful to employ the *Dirichlet-Laplace operator* whose construction we will give in what follows.

We consider the Hilbert space

$$H_0^1(\Omega)^3 = \overline{\mathcal{D}(\Omega)^3}^{(\cdot, \cdot)_{H^1}} \subseteq H^1(\Omega)^3$$

$$\text{with} \quad \langle u, v \rangle_{H^1} = \langle u, v \rangle + \sum_{i=1}^3 \langle \partial_i u, \partial_i v \rangle$$

whose dual is commonly designated by

$$H^{-1}(\Omega)^3 := (H_0^1(\Omega)^3)'$$

Now the bilinear form

$$\mathbf{a} : \begin{cases} H_0^1(\Omega)^3 \times H_0^1(\Omega)^3 \rightarrow \mathbb{R} \\ (u, v) \mapsto \sum_{i=1}^3 \langle \partial_i u, \partial_i v \rangle \end{cases}$$

induces the *Dirichlet-Laplace*³ operator given by

$$\Delta_D^\Omega : \begin{cases} H_0^1(\Omega)^3 \rightarrow H^{-1}(\Omega)^3 \\ v \mapsto -\mathbf{a}(\cdot, v) \end{cases}$$

4 Proposition. Δ_D^Ω is well-defined and bounded.

Proof.

$$\begin{aligned} |-\mathbf{a}(u, v)| &\leq \sum_{i=1}^3 |\langle \partial_i u, \partial_i v \rangle| \leq \sum_{i=1}^3 \|\partial_i u\| \|\partial_i v\| \\ &\leq \left(\sum_{i=1}^3 \|\partial_i u\|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^3 \|\partial_i v\|^2 \right)^{\frac{1}{2}} \\ &\leq \|u\|_{H^1} \|v\|_{H^1} \\ \Rightarrow \|-\mathbf{a}(\cdot, v)\| &\leq \|v\|_{H^1} \quad \text{and} \quad \|\Delta_D^\Omega\| \leq 1 \end{aligned}$$

We remark that the operator

$$\delta - \Delta_D^\Omega : \begin{cases} H_0^1(\Omega)^3 \rightarrow H^{-1}(\Omega)^3 \\ v \mapsto \delta \langle \cdot, v \rangle + \mathbf{a}(\cdot, v) \end{cases}$$

is an isomorphism for all $\delta > 0$. If the domain Ω is bounded, Poincaré's inequality shows that even Δ_D^Ω is an isomorphism.

4 Stokes operator

Now that we have a useful definition of the Laplace operator we need to “extend” the Helmholtz projection such that its domain contains $H^{-1}(\Omega)$.

The space

$$\mathcal{V} := H_0^1(\Omega)^3 \cap L_\sigma^2(\Omega)$$

is a closed subspace of $H_0^1(\Omega)^3$, and hence a Hilbert space in its own right (endowed with $\langle \cdot, \cdot \rangle_{H^1}$).

5 Proposition.

$$L_\sigma^2(\Omega) = \overline{\mathcal{V}}^{\|\cdot\|_{L^2}}$$

Proof. Since $\mathcal{D}_\sigma(\Omega) \subseteq \mathcal{V}$ this follows from proposition 2. \square

This already indicates that \mathcal{V} is still large enough a space to be useful. It will also prove useful later on.

Let $\tilde{J} : \mathcal{V} \rightarrow H_0^1(\Omega)^3$ be the canonical injection and let $\tilde{\mathbb{P}} : H^{-1}(\Omega) \rightarrow \mathcal{V}'$ be its adjoint. Clearly both are continuous mappings.

³This name reflects the fact that Δ_D^Ω is defined on the function space $H_0^1(\Omega)$ consisting of functions u satisfying the Dirichlet boundary condition $u|_{\partial\Omega} = 0$.

6 Proposition. $\tilde{\mathbb{P}}$ and \mathbb{P} are consistent in the following sense

$$\tilde{\mathbb{P}}(\langle \cdot, v \rangle) = \langle \cdot, \mathbb{P}v \rangle \quad \forall v \in L^2(\Omega)^3$$

Proof. For all $u \in \mathcal{V}$ we have

$$\tilde{\mathbb{P}}(\langle \cdot, v \rangle)(u) = \langle \tilde{J}u, v \rangle = \langle Ju, v \rangle = \langle u, \mathbb{P}v \rangle$$

\square

7 Proposition. The operator

$$A_0 : \begin{cases} \mathcal{V} \rightarrow \mathcal{V}' \\ v \mapsto -\tilde{\mathbb{P}}\Delta_D^\Omega \tilde{J}v \end{cases}$$

is bounded and self-adjoint in $(\mathcal{V}, \langle \cdot, \cdot \rangle_{H^1})$, and $A_0v = \mathbf{a}(\cdot, v)$ where \mathbf{a} is the bilinear form from section 3.

Proof. (1) A_0 is composed of bounded operators, hence bounded.

(2) For all $u, v \in \mathcal{V}$ we have

$$\begin{aligned} (A_0v)(u) &= (-\tilde{\mathbb{P}}\Delta_D^\Omega \tilde{J}v)(u) = -(\Delta_D^\Omega \tilde{J}v)(\tilde{J}u) \\ &= \mathbf{a}(\tilde{J}u, \tilde{J}v) = \mathbf{a}(u, v) \end{aligned}$$

(3) A_0 is defined on the entire Hilbert space \mathcal{V} (i.e. bounded), hence self-adjointness will follow from symmetry. But it is clear from (2) that A_0 is symmetric. \square

8 Definition. The *Stokes operator* is defined by

$$\begin{aligned} D(A) &:= \{v \in \mathcal{V} \mid A_0v \in L_\sigma^2(\Omega)\} \\ &:= \{v \in \mathcal{V} \mid \exists u \in L_\sigma^2(\Omega) : A_0v = \langle \cdot, u \rangle\} \end{aligned}$$

$$A : D(A) \rightarrow L_\sigma^2(\Omega), \quad v \mapsto u$$

9 Theorem. The Stokes operator is self-adjoint in $(L_\sigma^2(\Omega), \langle \cdot, \cdot \rangle)$, generates an analytic semigroup $(e^{-tA})_{t \geq 0}$, and satisfies

$$\begin{aligned} D(A) &= \{v \in \mathcal{V} \mid \exists \pi \in \mathcal{D}'(\Omega) : \nabla \pi \in H^{-1}(\Omega)^3 \\ &\quad \text{and } -\Delta_D^\Omega v + \nabla \pi \in L_\sigma^2(\Omega)\} \\ &:= \{v \in \mathcal{V} \mid \exists \pi \in \mathcal{D}'(\Omega) : \nabla \pi \in H^{-1}(\Omega)^3 \\ &\quad \text{and } \exists u \in L_\sigma^2(\Omega) : -\Delta_D^\Omega v + \nabla \pi = \langle \cdot, u \rangle\} \end{aligned}$$

$$Av = u$$

Proof. (1) By proposition 5 it follows that $\mathbf{b} := \mathbf{a}|_{\mathcal{V} \times \mathcal{V}}$ is a densely-defined bilinear form on $L_\sigma^2(\Omega)$. It is also:

- (i) accretive, as $\mathbf{b} \geq 0$,
- (ii) closed, since $D(\mathbf{b}) = \mathcal{V}$, $\|\cdot\|_{\mathbf{b}} = \|\cdot\|_{H^1}$ and we have seen before that $(\mathcal{V}, \langle \cdot, \cdot \rangle_{H^1})$ is complete,
- (iii) continuous in the sense that $|\mathbf{b}(u, v)| \leq C\|u\|_{\mathbf{b}}\|v\|_{\mathbf{b}}$; this is shown by proposition 4.

Under these conditions, the operator associated with the adjoint form \mathbf{b}^* is simply the adjoint of the operator associated with \mathbf{b} (see e.g. OUHABAZ [3]).

A is *by its very definition* the operator associated with \mathbf{b} . And \mathbf{b} is symmetric, hence $\mathbf{b}^* = \mathbf{b}$ and it follows that $A^* = A$, $D(A^*) = D(A)$, that is, A is self-adjoint.

(2) Another result from the theory of sesquilinear forms is that if A is the operator associated with a densely-defined bilinear form \mathfrak{b} satisfying (i)-(iii), $-A$ generates a (strongly continuous contraction) semi-group on $L^2_\sigma(\Omega)$ which is holomorphic in an open sector

$$\{z \in \mathbb{C} \mid 0 \neq z \text{ and } |\arg(z)| < \theta\}$$

In particular it is analytic on $\mathbb{R}_{>0}$.

(3) Let $Av = u$. Then

$$\begin{aligned} & -\tilde{\mathbb{P}}\Delta_D^\Omega v = \langle \cdot, u \rangle \stackrel{6}{=} \tilde{\mathbb{P}}(\langle \cdot, u \rangle) \\ \Rightarrow & \tilde{\mathbb{P}}(-\Delta_D^\Omega v - \langle \cdot, u \rangle) = 0 \\ \Rightarrow & -\Delta_D^\Omega v - \langle \cdot, u \rangle \in \ker(\tilde{\mathbb{P}}) \end{aligned}$$

It remains to show that the kernel consists of gradients:

$$\begin{aligned} \ker(\tilde{\mathbb{P}}) &= \{\phi \in H^{-1}(\Omega)^3 \mid \tilde{\mathbb{P}}\phi = 0\} \\ &= \{\phi \in H^{-1}(\Omega)^3 \mid \phi|_{\mathcal{V}} = 0\} \\ &\stackrel{2}{=} \{\phi \in H^{-1}(\Omega)^3 \mid \phi|_{\mathcal{D}_\sigma(\Omega)} = 0\} \\ &\stackrel{1}{=} \{\phi \in H^{-1}(\Omega)^3 \mid \exists \pi \in \mathcal{D}'(\Omega) : \phi = \nabla\pi\} \end{aligned}$$

□

5 General case

Let $p \in (1, \infty)$, $1 = \frac{1}{p} + \frac{1}{p'}$, $n \geq 2$ and $\Omega \subseteq \mathbb{R}^n$ be an arbitrary domain. Let $\langle u, v \rangle := \int_\Omega u \cdot v \, d\mu$. Define

$$\begin{aligned} \dot{W}^{1,p}(\Omega) &:= \{[u] = u + \mathbb{C} \mid u \in L^p_{loc}(\Omega) \\ &\quad \text{and } \nabla u \in L^p(\Omega)^n\} \\ \|[u]\|_{\dot{W}^{1,p}} &:= \|\nabla u\|_{L^p} \end{aligned}$$

This is a Banach space, as is its dual $\dot{W}^{1,p}(\Omega)'$. The gradient operator

$$\nabla_p : \dot{W}^{1,p}(\Omega) \rightarrow L^p(\Omega)^n, \quad u \mapsto \nabla u$$

is an isometry pretty much by definition of the norms. We isometrically identify $L^p(\Omega)$ and $(L^p(\Omega))'$, and it follows that the adjoint

$$\nabla'_p : L^p(\Omega)^n \rightarrow (\dot{W}^{1,p}(\Omega))'$$

is surjective with norm ≤ 1 . Define

$$\begin{aligned} G^p(\Omega) &:= \text{ran}(\nabla_p) \subseteq L^p(\Omega)^n \\ L^p_\sigma(\Omega) &:= \ker(\nabla'_p) \subseteq L^p(\Omega)^n \end{aligned}$$

Clearly this definition of G^p agrees with the one in the introduction. The following proposition shows that this also applies to L^p_σ .

10 Proposition.

$$L^p_\sigma(\Omega) = \overline{\mathcal{D}_\sigma(\Omega)}^{\|\cdot\|_{L^p}}$$

Proof. (1) Let $\phi \in \mathcal{D}_\sigma(\Omega)$. Then for all $u \in \dot{W}^{1,p'}(\Omega)$, we have

$$\begin{aligned} (\nabla'_{p'}\phi)(u) &= \langle \nabla_{p'}u, \phi \rangle = -\langle u, \text{div}(\phi) \rangle = 0 \\ \Rightarrow \phi &\in \ker(\nabla'_{p'}) = L^p_\sigma \end{aligned}$$

(2) In order to prove density it is sufficient to show that the values of a functional on $L^p_\sigma(\Omega)$ is already determined by its values on $\mathcal{D}_\sigma(\Omega)$. Equivalently, we show that the following assertion holds for all $v \in L^p(\Omega)^n \cong (L^p(\Omega)^n)'$:

$$\langle \cdot, v \rangle|_{\mathcal{D}_\sigma(\Omega)} = 0 \quad \Rightarrow \quad \langle \cdot, v \rangle|_{L^p_\sigma(\Omega)} = 0$$

By the assumption and fact 1, we have $v = \nabla_{p'}p$ for a $p \in L^{p'}(\Omega)$, hence

$$\forall u \in L^p_\sigma(\Omega) : \langle u, v \rangle = \langle u, \nabla_{p'}p \rangle = \langle \nabla'_{p'}u, p \rangle = 0 \quad \square$$

11 Theorem. We have the decomposition

$$L^p(\Omega)^n = L^p_\sigma(\Omega) \oplus G^p(\Omega)$$

if and only if the operator

$$N_p : \dot{W}^{1,p}(\Omega) \rightarrow (\dot{W}^{1,p'}(\Omega))', \quad N_p := \nabla'_{p'}\nabla_p$$

is bijective.

$$\begin{array}{ccc} & & (\dot{W}^{1,p'}(\Omega))' \\ & \nearrow^{\nabla'_{p'}} & \uparrow N_p \\ L^p(\Omega)^n & & \dot{W}^{1,p}(\Omega) \\ & \nwarrow_{\nabla_p} & \end{array}$$

In that case,

$$\mathbb{P}_p : L^p(\Omega)^n \rightarrow L^p_\sigma(\Omega), \quad \mathbb{P}_p := \text{id} - \nabla_p N_p^{-1} \nabla'_{p'}$$

is a projection with kernel $G^p(\Omega)$. It is called the Helmholtz projection in $L^p(\Omega)$.

Proof. (\Rightarrow) Since the sum is direct, we have

$$\ker(\nabla'_{p'}) \cap \text{ran}(\nabla_p) = \{0\}$$

and N_p is injective. Also,

$$\begin{aligned} \ker(\nabla'_{p'}) + \text{ran}(\nabla_p) &= L^p(\Omega)^n \\ \Rightarrow \text{ran}(N_p) &= \nabla'_{p'}(\text{ran}(\nabla_p)) = \nabla'_{p'}(L^p(\Omega)^n) = \text{ran}(\nabla'_{p'}) \end{aligned}$$

and N_p is surjective since $\nabla'_{p'}$ is so.

(\Leftarrow) If N_p is bijective, its inverse is bounded by the open mapping theorem, hence \mathbb{P}_p is bounded. Clearly,

$$\mathbb{P}_p|_{L^p_\sigma(\Omega)} = \text{id}$$

and

$$\begin{aligned} \mathbb{P}_p v = 0 &\Rightarrow v \in \text{ran}(\nabla_p) = G^p(\Omega) \\ v = \nabla_p u \in G^p(\Omega) &\Rightarrow \mathbb{P}_p v = v - \underbrace{\nabla_p N_p^{-1} \nabla'_{p'} u}_{=\nabla_p u = v} = 0 \\ \Rightarrow \ker(\mathbb{P}_p) &= G^p(\Omega) \end{aligned}$$

Hence, the sum $L^p_\sigma(\Omega) + G^p(\Omega)$ is direct, and since every $v \in L^p(\Omega)^n$ can be written as

$$v = \underbrace{\mathbb{P}_p v}_{\in L^p_\sigma(\Omega)} + \underbrace{(v - \mathbb{P}_p v)}_{\in \ker(\mathbb{P}_p) = G^p(\Omega)}$$

it is entire space. □

Note that for $\phi \in \mathcal{D}(\Omega)$, $v \in \dot{W}^{1,p'}(\Omega)$ we have

$$(N_p\phi)(v) = \langle \nabla_p\phi, \nabla_{p'}v \rangle = \langle -\Delta\phi, v \rangle$$

hence $-N_p$ can be interpreted as a weak formulation of the Laplace operator.

From chapter 2, we see that for $n = 3$, N_2 is always bijective and \mathbb{P}_2 is just the orthogonal projection \mathbb{P} onto $L^2_\sigma(\Omega)$.

Finally we remark that \mathbb{P}_p can be extended in a similar way as \mathbb{P} was extended to $\tilde{\mathbb{P}}$ in chapter 4, see HAAK and KUNSTMANN [1] for details.

References

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