

# THE GEOMETRIC GROUP OF THE BAUM-CONNES CONJECTURE FOR DISCRETE GROUPS

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**Notation and Conventions.** Let  $\Gamma$  be a countable discrete group. We will assume that all spaces are metrizable (in particular, Hausdorff and paracompact).

## 1. PROPER ACTIONS

**Proper  $\Gamma$ -spaces.** Let  $X$  be a  $\Gamma$ -space.  $X$  is called *proper* if the orbit space  $\Gamma \backslash X$  is Hausdorff and every point  $x \in X$  has a  $\Gamma$ -invariant open neighborhood  $U$  with a  $\Gamma$ -map  $U \rightarrow \Gamma/H$  for some *finite* subgroup  $H \subseteq \Gamma$ .

**Example 1.** Add Łukasz' examples...

The following lemma is central in proving important properties about proper  $\Gamma$ -spaces:

**Technical Lemma 2** ([BCH94, Appendix 1, Lemma]). *Let  $X$  be a proper  $\Gamma$ -space. Then we can find a countable partition of unity  $(\sigma_k)$  of  $\Gamma$ -invariant functions and  $\Gamma$ -maps*

$$\text{supp}(\sigma_k) \xrightarrow{\phi_k} \coprod_{H \subseteq \Gamma \text{ finite}} \Gamma/H$$

We can use it to show that every proper  $\Gamma$ -space is also proper in the usual sense:

**Lemma 3.** *Let  $X$  is a proper  $\Gamma$ -space and  $K, L \subseteq X$  compact. Then the set*

$$\{\gamma \in \Gamma : \gamma K \cap L \neq \emptyset\}$$

*is finite.*

*Proof.* From the Technical Lemma we get a  $\Gamma$ -map

$$\phi : X \rightarrow \coprod_{H \subseteq \Gamma \text{ finite}} \Gamma/H, \quad x \mapsto \sum \sigma_k(x) \phi_k(x)$$

Noting that the target space is discrete we see that the images of  $K$  and  $L$  under this map are finite. In particular, they are contained in already finitely many summands  $\Gamma/H_i$ . Hence the above set contains at most

$$\#\phi(K) \cdot \#\phi(L) \cdot \max_i |H_i|$$

group elements. □

The preceding lemma in turn can be used to prove many other results about proper spaces.

**Lemma 4.** *Let  $X$  be a proper  $\Gamma$ -space and  $f \in C_c(X)$ . Then*

$$\left\| \sum_{\gamma \in \Gamma} \gamma f \right\|_{\infty} < \infty$$

where the sum is understood as a pointwise limit.

*Proof.* If  $x \in \text{supp}(\gamma_0 f)$  for some  $\gamma_0 \in \Gamma$  then

$$\left| \sum_{\gamma \in \Gamma} (\gamma f)(x) \right| = \left| \sum_{\gamma \in \Gamma} f(\gamma^{-1}x) \right| \leq \|f\|_{\infty} \cdot \#\{\gamma \in \Gamma : \gamma^{-1}x \in \text{supp}(f)\}$$

But

$$\begin{aligned} & \#\{\gamma \in \Gamma : \gamma^{-1}x \in \text{supp}(f)\} \\ & \leq \#\{\gamma \in \Gamma : \gamma^{-1} \text{supp}(\gamma_0 f) \cap \text{supp}(f) \neq \emptyset\} \\ & \leq \#\{\gamma \in \Gamma : \gamma^{-1} \text{supp}(f) \cap \text{supp}(f) \neq \emptyset\} < \infty \end{aligned}$$

independent of  $x$ . □

A proper  $\Gamma$ -space  $\underline{E}\Gamma$  is called *universal* if for every other proper  $\Gamma$ -space  $X$  there exists a unique  $\Gamma$ -map  $X \rightarrow \underline{E}\Gamma$  up to  $\Gamma$ -homotopy. Clearly, any two such spaces are unique up to  $\Gamma$ -homotopy equivalence.

**Proposition 5.** *The space*

$$\underline{E}\Gamma := \{f : \Gamma \rightarrow [0, 1] : f \text{ finitely supported, } \sum_{\gamma \in \Gamma} f(\gamma) = 1\}$$

*equipped with the supremum metric and the obvious left  $\Gamma$ -action is a universal proper  $\Gamma$ -space.*

*Proof. Properness:* It follows from the finite support condition that the metric “descends” to the quotient space. Hence it is metrizable and, in particular, Hausdorff. Now let  $f \in \underline{E}\Gamma$ . Since it has finite support we observe that

$$R := \inf\{\|f - \gamma f\|_{\infty} : \gamma \in \Gamma \setminus \text{Stab}(f)\} > 0$$

If we choose  $\epsilon \in (0, \frac{R}{2})$  then for every element in the open  $\Gamma$ -invariant neighborhood

$$U := \{g \in \underline{E}\Gamma : \exists \gamma_g \in \Gamma : \|g - \gamma_g f\| < \epsilon\}$$

the associated  $\gamma_g$  is uniquely define modulo  $\text{Stab}(f)$ . Thus we get a  $\Gamma$ -map

$$U \rightarrow \Gamma / \text{Stab}(f), \quad g \mapsto \gamma_g \text{Stab}(f)$$

*Universality:* Let  $X$  be any proper  $\Gamma$ -space. Evidently, any two  $\Gamma$ -maps  $X \rightarrow \underline{E}\Gamma$  are  $\Gamma$ -homotopic (using convexity of the latter space). Thus it suffices to construct a single such map: Let us invoke the Technical Lemma again and define the  $\Gamma$ -map

$$\phi : \coprod_{H \subseteq \Gamma \text{ finite}} \Gamma/H \rightarrow \underline{E}\Gamma, \quad \gamma H \mapsto \frac{1}{\#H} 1_H$$

sending a  $H$ -coset to the uniform probability distribution on  $H$ . Then

$$X \rightarrow \underline{E}\Gamma, \quad x \mapsto \sum_k \sigma_k(x) (\phi \circ \phi_k(x))$$

is a  $\Gamma$ -map of the desired form. □

**$\Gamma$ -compactness.** A proper  $\Gamma$ -space is called  $\Gamma$ -compact if  $\Gamma \backslash X$  is compact.

**Lemma 6.** *Every  $\Gamma$ -compact proper  $\Gamma$ -space is locally compact.*

*Proof. Missing. . .*

□

## 2. EQUIVARIANT $K$ -HOMOLOGY

The *equivariant  $K$ -homology* groups of a locally compact  $\Gamma$ -space  $X$  are defined as follows in terms of Kasparov's equivariant  $KK$ -theory (cf. [Kas88]):

$$K_*^\Gamma(X) := KK_*^\Gamma(C_0(X), \mathbb{C})$$

We cannot apply this construction to an arbitrary proper  $\Gamma$ -space  $Y$  since it will in general *not* be locally compact. However, all its  $\Gamma$ -compact subspaces are locally compact by Lemma 6. Hence we will define its *equivariant  $K$ -homology* as the direct limit

$$RK_*^\Gamma(Y) := \varinjlim_{X \subseteq Y \text{ } \Gamma\text{-compact}} K_*^\Gamma(X)$$

The arrows of the directed system are induced by inclusions  $\Gamma$ -compact subspaces  $X \subseteq X'$ . In order for this to make sense we have to show that any such  $X$  is closed in  $X'$ . But this follows readily from commutativity of the diagram

$$\begin{array}{ccc} X & \xrightarrow{\subseteq} & X' \\ \downarrow & & \downarrow \pi \\ \Gamma/X & \xrightarrow{\subseteq} & \Gamma/X' \end{array}$$

$$(X = \pi^{-1}(\Gamma/X)).$$

In particular, we can apply this construction to the universal proper  $\Gamma$ -space  $\underline{E}\Gamma$ . The resulting groups  $RK_*^\Gamma(\underline{E}\Gamma)$  are the *geometric groups* of the Baum-Connes conjecture for the discrete group  $\Gamma$ .

**Cycles.** Recall that cycles for  $K_*^\Gamma(X)$  are given by triples  $(H, \pi, F)$  containing the following data

- (i)  $H$  is a Hilbert space with a unitary  $\Gamma$ -representation
- (ii)  $\pi : C_0(X) \curvearrowright H$  *covariant* representation, i.e.  $\pi(\gamma f) = \gamma \pi(f) \gamma^{-1}$
- (iii)  $F$  is a self-adjoint bounded operator on  $H$

such that

$$[\pi(f), F], \quad \pi(f)(F^2 - 1), \quad \pi(f)[\gamma, F]$$

are compact for all  $f \in C_0(X)$ ,  $\gamma \in \Gamma$ . In the even case, we also require  $H$  to be equipped with a  $\mathbb{Z}_2$ -grading such that  $\Gamma$  and  $C_0(X)$  act as even operators and  $F$  is odd.

**Simplifications.** Since a cycle  $(H, \pi, F)$  is homotopic to its compression to the orthogonal complement of the common kernel of the  $C_0(X)$ -action we can (and will) always assume that  $\pi$  is a *non-degenerate* representation. If  $X$  is a  $\Gamma$ -compact proper space then we can assume even more:

**Proposition 7.** *Let  $X$  be a  $\Gamma$ -compact proper space. Then any cycle  $(H, \pi, F)$  for  $K_*^\Gamma(X)$  is a compact perturbation of a cycle  $(H, \pi, G)$  where  $F'$  is  $\Gamma$ -equivariant and properly supported, i.e.*

$$\forall g \in C_c(X) : \exists f \in C_c(X) : G\pi(g) = \pi(f)G\pi(g)$$

The main ingredient to proving this proposition is the following construction:

**Lemma 8.** *In the situation of Proposition 7 let  $k \in C_c(X)$  a real-valued function. Then*

$$A_k(F) := \sum_{\gamma \in \Gamma} \gamma \pi(k) F \pi(k) \gamma^{-1}$$

*converges strongly to a bounded operator on  $H$  (with  $\|A_k(F)\| \leq C_k \|F\|$ ). Furthermore,  $A_k(F)$  is  $\Gamma$ -equivariant and properly supported.*

*Proof.* For showing strong convergence we may assume that  $F \geq 0$ . Then, since  $\Gamma \curvearrowright H$  by unitaries and  $\pi(\gamma k)$  is self-adjoint, every summand

$$\gamma \pi(k) F \pi(k) \gamma^{-1} = \pi(\gamma k) \gamma F \gamma^{-1} \pi(\gamma k)$$

is also positive. Hence by Lemma 9 it suffices to establish a uniform norm bound for the partial sums: By conjugating  $F \leq \|F\|$  we find that

$$\pi(\gamma k) \gamma F \gamma^{-1} \pi(\gamma k) \leq \pi(\gamma k^2) \|F\|,$$

hence for every finite subset  $\Gamma' \subseteq \Gamma$  we get the estimate

$$\left\| \sum_{\gamma \in \Gamma'} \pi(\gamma k^2) \right\| \leq \left\| \sum_{\gamma \in \Gamma'} \gamma k^2 \right\|_\infty \leq \left\| \sum_{\gamma \in \Gamma} \gamma k^2 \right\|_\infty =: C_k \stackrel{4}{\leq} \infty$$

We conclude that  $A_k(F)$  exists as a strong limit and that its norm is bounded by  $C_k \|F\|$ . Since  $A_k(F)$  is positive this holds for any self-adjoint operator  $F$ .

It remains to show that  $A_k(F)$  is  $\Gamma$ -equivariant and properly supported. The former is evident from the definition. For the latter, let  $g \in C_c(X)$ . We have

$$A_k(F)\pi(g) = \sum_{\gamma \in \Gamma} \pi(\gamma k) \gamma F \pi(k \cdot (\gamma^{-1}g)) \gamma^{-1}$$

As  $X$  is a proper  $\Gamma$ -space, the set

$$S := \{\gamma \in \Gamma : k \cdot (\gamma^{-1}g) \neq 0\} = \{\gamma \in \Gamma : \text{supp}(k) \cap \gamma^{-1} \text{supp}(g) \neq \emptyset\}$$

is finite. Hence  $V := \bigcup_{\gamma \in S} \text{supp}(\gamma k)$  is compact and we can find a function  $f \in C_c(X)$  equal to 1 on  $V$ . Any such function satisfies  $\pi(f)A_k(F)\pi(g) = A_k(F)\pi(g)$ .  $\square$

*Proof of Proposition 7.* By [Bou63, 7.2.4, Proposition 8] there exists a function  $0 \leq k \in C_c(X)$  with

$$\sum_{\gamma \in \Gamma} k^2(\gamma x) = 1 \quad (\forall x \in X)$$

Note that its partial sums define an approximate unit in  $C_0(X)$ . Thus it follows from the fact that we assumed  $\pi$  to be non-degenerate that

$$\sum_{\gamma \in \Gamma} \gamma \pi(k^2) \gamma^{-1} F = F$$

(strong convergence). Let us show that  $G := A_k(F)$  is a compact perturbation of the operator  $F$ . Indeed, for every  $f$  in the dense subset  $C_c(X) \subseteq C_0(X)$  we get

$$\begin{aligned} & \pi(f)(F - G) \\ &= \sum_{\gamma \in \Gamma} \pi(f) \gamma \pi(k^2) \gamma^{-1} F - \gamma \pi(k) F \pi(k) \gamma^{-1} \\ &= \sum_{\gamma \in \Gamma} \pi(f) \gamma \pi(k) (\pi(k) \gamma^{-1} F - F \pi(k) \gamma^{-1}) \\ &= \sum_{\gamma \in \Gamma} \pi(f) \gamma \pi(k) (\pi(k) [\gamma^{-1}, F] + [\pi(k), F] \gamma^{-1}) \\ &= \sum_{\gamma \in \Gamma} \gamma \pi((\gamma^{-1} f) \cdot k) (\pi(k) [\gamma^{-1}, F] + [\pi(k), F] \gamma^{-1}) \end{aligned}$$

Observe that every summand is compact. Thus it follows from

$$\#\{\gamma \in \Gamma : \text{supp}(\gamma^{-1} f) \cap (\text{supp } k) \neq \emptyset\} < \infty$$

being finite that  $\pi(f)(F - G)$  is in fact a finite sum of compact operators, hence compact.  $\square$

#### APPENDIX A. FUNCTIONAL ANALYSIS

**Lemma 9.** *Any increasing sequence  $(A_n) \subseteq B(H)$  of positive operators which is uniformly norm-bounded is strongly convergent (to a positive operator).*

*Proof.* For every  $x \in H$ ,  $(\langle A_n x, x \rangle)$  is an increasing sequence of positive numbers bounded above by  $\sup \|A_n\| \cdot \|x\|^2$ , hence convergent. By the polarization formula it follows that  $(\langle A_n x, y \rangle)$  is convergent for every  $x, y \in H$ . Hence  $A_n$  converges weakly to some  $A \geq A_n$ . But then

$$\|(A - A_n)^{\frac{1}{2}} x\|^2 = \langle (A - A_n)^{\frac{1}{2}} x, (A - A_n)^{\frac{1}{2}} x \rangle = \langle (A - A_n) x, x \rangle \rightarrow 0,$$

i.e.  $(A - A_n)^{\frac{1}{2}} \rightarrow 0$  strongly and hence also  $A_n \rightarrow A$  strongly.  $\square$

#### REFERENCES

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