A short remark on convergence in stochastics

1 Theorem. *Let $(X_n), (Y_n), X, Y$ be random variables on the same probability space, and let $c \in \mathbb{R}$. Then:*

(i) $X_n \xrightarrow{\mathbb{P}} X,\ Y_n \xrightarrow{\mathbb{P}} Y \quad \Rightarrow \quad X_n + Y_n \xrightarrow{\mathbb{P}} X + Y$
(same for almost sure convergence)

(ii) $X_n \xrightarrow{\mathcal{D}} c \quad \Rightarrow \quad X_n \xrightarrow{\mathbb{P}} c$

2 Corollary. *(i) cannot hold for convergence in distribution.*

Proof. Otherwise we could lift convergence in distribution to convergence in probability:

$$
\begin{aligned}
&X_n \xrightarrow{\mathcal{D}} X \\
\overset{(i)}{\Rightarrow} &X_n - X \xrightarrow{\mathcal{D}} 0 \\
\overset{(ii)}{\Rightarrow} &X_n - X \xrightarrow{\mathbb{P}} 0 \\
\overset{(i)}{\Rightarrow} &X_n \xrightarrow{\mathbb{P}} X
\end{aligned}
$$

But we know that the former is strictly weaker than the latter. □

3 Corollary. *(ii) cannot apply to convergence in probability together with almost sure convergence.*

Proof. By the same argument we could lift convergence in probability to almost sure convergence. Contradiction. □