A SURVEY OF SPIN^c STRUCTURE

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Conventions. All differential geometric objects are assumed to be smooth.

Part I. Preliminaries

1. Clifford algebras

In this section we will give a very brief overview of the theory of Clifford algebras (loosely based on the much more detailed treatment in [LM89, §1]).

Clifford algebras. Let V be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The *Clifford algebra* of V with respect to a quadratic form q is the $\mathbb{K}$-algebra

$$\operatorname{Cliff}(V,q) := \bigoplus_{n \in \mathbb{N}_0} V^{\otimes n} / \{v \otimes v + q(v) : v \in V\}$$

Note that we can consider $V \subseteq \operatorname{Cliff}(V,q)$ since the canonical projection is injective on V. Also note that $v \in V$ is invertible in the Clifford algebra if $q(v) \neq 0$.

Proposition 1 (Universal property)**.** *Every linear map $f : V \to A$ into a unital $\mathbb{K}$-algebra A satisfying $f(v)^2 = -q(v)1$ extends to an algebra homomorphism $\operatorname{Cliff}(V,q) \to A$.*

If $e_1, \dots, e_n$ is a basis of V, the monomials $e_{i_1} \cdots e_{i_k}$ ($i_1 < \dots < i_k$, $0 \leqslant k \leqslant n$) form a basis of $\operatorname{Cliff}(V,q)$. In particular,

$$\dim \operatorname{Cliff}(V,q) = 2^{\dim V}$$

and we can *grade* the Clifford algebra such that its even (odd) part is generated by the even (odd) monomials. Evidently, this is the grading given by the *grading endomorphism* α extending $v \mapsto -v$ on V.

We can equip $\operatorname{Cliff}(V,q)$ with an inner product such that the action of unit vectors is orthogonal, i.e. $\langle e \cdot v, e \cdot w \rangle = \langle v, w \rangle$ if $q(e) = 1$. In particular, the adjoint of left multiplication with e is left multiplication with its $-e$, and more generally the adjoint of left multiplication with an arbitrary element of the Clifford algebra is given by left multiplication with some other element. Hence we can define an involution on $\operatorname{Cliff}(V,q)$ satisfying

$$\langle x \cdot v, w \rangle = \langle v, x^* \cdot w \rangle \quad (\forall x, v, w)$$

In particular, left multiplication is a graded $*$-representation.

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O, SO, Pin, Spin. As usual, we define the *orthogonal* and *special orthogonal* subgroups of invertible transformations of V:

$$\begin{aligned} O(V,q) &:= \{x \in GL(V) : x^*q = q\} \subseteq GL(V) \\ SO(V,q) &:= \{x \in O(V,q) : \det(x) = 1\} \end{aligned}$$

Example 2. For any vector $v \in V$ with $q(v) \neq 0$, *reflection* across $v^\perp$

$$\mathrm{refl}_v : w \mapsto w - 2\frac{q(v,w)}{q(v)}v$$

is an orthogonal transformation.

Note that, by the universal property, transformations in the orthogonal group extend to endomorphisms of the Clifford algebra and we get an action $O_n \circlearrowright \mathrm{Cliff}(V,q)$.

Let us also define the *pin* and *spin* subgroups of the unit group of the Clifford algebra:

$$\begin{aligned} \mathrm{Pin}(V,q) &:= \langle\{v \in V : q(v) = \pm 1\}\rangle \subseteq \mathrm{Cliff}(V,q)^x \\ \mathrm{Spin}(V,q) &:= \mathrm{Pin}(V,q) \cap \mathrm{Cliff}(V,q)_+ \end{aligned}$$

The *twisted adjoint representation* of the pin group is

$$\widetilde{\mathrm{ad}} : \mathrm{Pin}(V,q) \circlearrowright \mathrm{Cliff}(V,q),\ x \cdot y := \alpha(x)yx^{-1}$$

Clearly, the twisted adjoint representation agrees with the usual adjoint representation on $\mathrm{Spin}(V,q)$.

If q is non-degenerate then there are convenient representations of the groups we have just defined:

Proposition 3. *If q is non-degenerate then*

$$\begin{aligned} O(V,q) &= \{\mathrm{refl}_{v_1} \circ \cdots \circ \mathrm{refl}_{v_k} : v_i \in V, q(v_i) = \pm 1\}, \\ SO(V,q) &= \{\mathrm{refl}_{v_1} \circ \cdots \circ \mathrm{refl}_{v_k} : v_i \in V, q(v_i) = \pm 1, k \text{ even}\}, \\ Pin(V,q) &= \{v_1 \cdots v_k : v_i \in V, q(v_i) = \pm 1\}, \\ Spin(V,q) &= \{v_1 \cdots v_k : v_i \in V, q(v_i) = \pm 1, k \text{ even}\} \end{aligned}$$

Now a quick calculation shows that $\widetilde{\mathrm{ad}}(v)|_V = \mathrm{refl}_v$. Together with the preceding this proves part of the following proposition.

Proposition 4. *We have short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \sqrt{\pm 1} \arrow[r] & Pin(V,q) \arrow[r, "\widetilde{\mathrm{ad}}(\cdot)|_V"] & O(V,q) \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \sqrt{\pm 1} \arrow[r] & Spin(V,q) \arrow[r, "\widetilde{\mathrm{ad}}(\cdot)|_V"] & SO(V,q) \arrow[r] & 0 \end{tikzcd}$$

In particular, the (s)pin group is the $|\sqrt{\pm 1}|$-fold cover of the (special) orthogonal group.

Corollary 5. *The (standard or twisted) adjoint representation $Spin(V,q) \circlearrowright \mathrm{Cliff}(V,q)$ descends to the standard representation $SO(V,q) \circlearrowright \mathrm{Cliff}(v,q)$.*

Pin^c, Spin^c. Now assume that V is a *real* vector space. Then $\sqrt{\pm 1} = \{\pm 1\} \cong \mathbb{Z}_2$ and $\mathrm{Cliff}(V, q) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Cliff}(V \otimes_{\mathbb{R}} \mathbb{C}, q_{\mathbb{C}})$. The *complex pin* and *spin* groups are

$$\begin{aligned}\mathrm{Pin}^c(V, q) &:= \mathrm{Pin}(V, q) \times_{\mathbb{Z}_2} U(1)\\ \mathrm{Spin}^c(V, q) &:= \mathrm{Spin}(V, q) \times_{\mathbb{Z}_2} U(1)\end{aligned}$$

where the respective $\mathbb{Z}_2$-action is given by multiplication with -1. It is obvious that we can consider

$$\mathrm{Spin}^c(V, q) \subseteq \mathrm{Pin}^c(V, q) \subseteq \mathrm{Cliff}(V, q) \otimes_{\mathbb{R}} \mathbb{C}$$

The following proposition holds similarly for Pin^c.

Proposition 6. *We have short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & U(1) \arrow[r] & Spin^c(V,q) \arrow[r] & SO(V,q) \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & Spin^c(V,q) \arrow[r, "\xi_0"] & SO(V,q) \times U(1) \arrow[r] & 0 \end{tikzcd}$$

where $\xi_0 : [x, \lambda] \mapsto (\widetilde{\mathrm{ad}}(x)|_V, \lambda^2)$. *In particular, the complex spin group is the two-fold cover of* $SO(V, q) \times U(1)$.

Standard Clifford algebras. The real and complex *standard Clifford algebras* are defined as follows:

$$\mathbb{R}_{r,s} := \mathrm{Cliff}(\mathbb{R}^{r+s}, \sum_{i=1}^{r} x_i^2 - \sum_{i=r+1}^{s} x_i^2), \quad \mathbb{C}_n := \mathrm{Cliff}(\mathbb{C}^n, \sum_{i=1}^{n} z_i^2)$$

Up to isomorphism, these are the only finite-dimensional Clifford algebras over $\mathbb{R}$ and $\mathbb{C}$ with non-degenerate quadratic form. Let us write $\mathbb{R}_n := \mathbb{R}_{n,0}$. Then clearly $\mathbb{R}_n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}_n$.

We denote the (special) orthogonal and (complex) (s)pin groups of $\mathbb{R}_n$ by O_n, SO_n, Pin_n, Spin_n and Spin_n^c. Let us briefly state the previous results which are most important for us in this special situation.

Proposition 7.

$$\begin{aligned} O_n &= \{\mathrm{refl}_{v_1} \circ \cdots \circ \mathrm{refl}_{v_k} : v_i \in V, \|v_i\| = 1\},\\ SO_n &= \{\mathrm{refl}_{v_1} \circ \cdots \circ \mathrm{refl}_{v_k} : v_i \in V, \|v_i\| = 1, k \textit{ even}\},\\ Pin_n &= \{v_1 \cdots v_k : v_i \in V, \|v_i\| = 1\},\\ Spin_n &= \{v_1 \cdots v_k : v_i \in V, \|v_i\| = 1, k \textit{ even}\} \end{aligned}$$

Proposition 8. *We have short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & Spin_n \arrow[r, "\widetilde{\mathrm{ad}}"] & SO_n \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & U(1) \arrow[r] & Spin_n^c \arrow[r] & SO_n \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & Spin_n^c \arrow[r, "\xi_0"] & SO_n \times U(1) \arrow[r] & 0 \end{tikzcd}$$

For $n \geqslant 3$, *Spin*$_n$ *is the* universal cover *of* SO_n.

Complex representation theory. The main result of the classification theory of complex Clifford algebras is the following:

Proposition 9. *$\mathbb{C}_{2k} \cong M_{2^k}(\mathbb{C})$ and $\mathbb{C}_{2k+1} \cong M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C})$.*

Since the single irreducible representation of $M_n(\mathbb{C})$ is given by matrix multiplication with $\mathbb{C}^n$ we have the following corollary.

Corollary 10. *$\mathbb{C}_{2k}$ has a single irreducible representations (of dimension 2^k). $\mathbb{C}_{2k+1}$ has two irreducible representations (of dimension 2^k) which agree when restricted to $Spin^c_{2k+1}$.*

We can also consider *graded* representations.

Proposition 11. *The category of $\mathbb{Z}_2$-graded representations of $\mathbb{C}_{n+1}$ is equivalent to the category of ungraded representations of $\mathbb{C}_n$.*

Proof. We can consider the even part as a representation of $\mathbb{C}_{n+1,+} \cong \mathbb{C}_n$. Conversely, if S is an ungraded representation of $\mathbb{C}_n$ then $\mathbb{C}_{n+1} \otimes_{\mathbb{C}_{n+1,+}} S$ is a $\mathbb{Z}_2$-graded representation of $\mathbb{C}_{n+1}$. The constructions are inverse to each other. ☐

Corollary 12. *$\mathbb{C}_{2k}$ has two graded irreducible representations (of dimension 2^k) whose only difference is that the even and odd parts are swapped. Both representations agree when restricted to $Spin^c_{2k}$.*

$\mathbb{C}_{2k+1}$ has a single graded irreducible representation (of dimension 2^{k+1}).

In the following example we analyze the *even-dimensional* case.

Example 13. Let Δ_{2k} be the unique ungraded irreducible representation of $\mathbb{C}_{2k}$. The action of the *complex volume element* $\omega := i^k e_1 \cdots e_n$ is a central involution, i.e. a grading operator. It endows Δ_{2k} with its *standard grading.*

By using the negative volume element $-\omega$, or equivalently by swapping the even and odd parts, we get another graded irreducible representation. It is evident that these representations are inequivalent: the action of volume element changes sign when passing from one to the other.

A representation $\rho : A \to \operatorname{End}(S)$ of a *-algebra A is called a **-representation* if S is equipped with an inner product so that ρ is a *-homomorphism. In the graded case we also require that the inner product respects the grading (i.e. $S_+ \perp S_-$).

Proposition 14. *Every (ungraded or graded) $\mathbb{C}_n$-representation can be made a *-representation by choice of proper inner product.*

Proof. Choose an arbitrary inner product such that the even and odd parts are orthogonal. By averaging over the *Clifford group*

$$E_n := \{\pm e_1^{i_1} \cdots e_n^{i_n} : i_1, \ldots, i_n \in \{0, 1\}\}$$

which is a *finite* group of order 2^{n+1} we get an E_n-invariant inner product which still respects the grading. Since E_n generates $\mathbb{C}_n$ as an algebra, E_n-invariance already guarantees that we have a *-representation. ☐

Proposition 15. *If $n = 2k$ is even then any irreducible representation on a vector space S is an isomorphism of algebras (also in case of graded and *-algebras). In particular, $\mathbb{C}_n \cong S \hat{\otimes} S^*$.*

Proof. Every irreducible representation is injective. Surjectivity follows from comparing dimensions:

$$\dim \mathbb{C}_{2k} = 2^{2k} = (\dim S)^2 = \dim \operatorname{End}(S)$$

□

There are two standard *-representations of $\mathbb{C}_n$ on itself: The *full spinor representation* given by left multiplication (which is graded), and the *adjoint representation* we have seen before (which is not graded).

Clifford algebra bundles. The *Clifford algebra bundle* of a Riemannian vector bundle V over a manifold M and its *complexification* are the associate bundles

$$\begin{aligned} \operatorname{Cliff}(V) &:= P_{O_n}(V) \times_{O_n} \mathbb{R}_n \\ \operatorname{Cliff}^{\mathbb{C}}(V) &:= P_{O_n}(V) \times_{O_n} \mathbb{C}_n \end{aligned}$$

(using the standard actions $O_n \circlearrowright \mathbb{R}_n \subseteq \mathbb{C}_n$). Clearly, each fiber of $\operatorname{Cliff}(V)$ has the structure of a Clifford algebra.

If (E_i) is a local orthonormal frame (E_i) over $U \subseteq M$, we can define a bundle morphism by linear extension of

$$V|_U \to \operatorname{Cliff}(V)|_U, E_i|_u \mapsto [(E_i|_u), e_i]$$

In fact, this map does not depend on the choice of local orthonormal frame, hence extends to all of V. It is also easily seen to be injective. Thus we can consider $V \subseteq \operatorname{Cliff}(V)$.

Proposition 16. *Each fiber of the Clifford algebra bundle is the Clifford algebra of the respective fiber (endowed with the inner product given by the Riemannian metric g):*

$$\begin{aligned} \operatorname{Cliff}(V)_m &\cong \operatorname{Cliff}(V_m, g_m) \\ \operatorname{Cliff}^{\mathbb{C}}(V)_m &\cong \operatorname{Cliff}(V_m, g_m) \otimes_{\mathbb{R}} \mathbb{C} \end{aligned}$$

Proof. This is evident because the above inclusion map sends an orthonormal basis with respect to g onto an orthonormal basis with respect to the standard inner product of $\mathbb{R}^n$ (which is just the quadratic form used to define $\mathbb{R}_n$). □

Proposition 17. *The complexified Clifford algebra bundle is indeed the complexification of the Clifford algebra bundle:*

$$\operatorname{Cliff}^{\mathbb{C}}(V) = \operatorname{Cliff}(V) \otimes_{\mathbb{R}} \mathbb{C}$$

We can further reduce the structure group if V is oriented:

Proposition 18. *Suppose V is an* oriented *Riemannian vector bundle. Then*

$$\begin{aligned} \operatorname{Cliff}(V) &= P_{SO_n}(V) \times_{SO_n} \mathbb{R}_n \\ \operatorname{Cliff}^{\mathbb{C}}(V) &= P_{SO_n}(V) \times_{SO_n} \mathbb{C}_n \end{aligned}$$

2. Multigrading

Multigraded bundles; [HR00, App. A] TODO

3. CONNECTIONS

Connections on vector bundles. Definition of connections (including $\Gamma^\infty(E) \to \Gamma^\infty(T^*M \otimes E)$ picture), compatibility with metric; Levi-Civita connection; [Roe98, §1]

TODO

Connections on principal bundles. Definition of connections; connection on $P_{GL_n}(V)$ ($P_{O_n}(V)$ etc.) vs. connection on V (compatible with metric etc.); holonomy of associated bundles given by left action; [Roe98, §2]

TODO

4. COHOMOLOGY AND CHARACTERISTIC CLASSES

Let X be a paracompact Hausdorff space and G a topological group.

Cohomology and principal bundles. The set of equivalence classes of principal G-bundles over X is naturally represented by the first Čech cohomology group (or *set* for nonabelian G) of X with coefficients in G (see [Hir78, §3], [LM89, App. A]):

$$\mathrm{Prin}_G(X) \cong \check{H}^1(X; G)$$

Chern class. The *first Chern class* is the coboundary map

$$c_1 : \mathrm{Vect}_1^{\mathbb{C}}(X) \cong \mathrm{Prin}_{U_1}(X) \cong H^1(X; U_1) \overset{\cong}{\to} H^2(X; \mathbb{Z})$$

associated to the extension $\mathbb{R}$ of $\mathbb{Z}$ via U_1. It is an isomorphism, and it agrees with the ordinary definition in the special case of complex line bundles.

Stiefel-Whitney classes. The *second Stiefel-Whitney class* is the coboundary map

$$w_2 : H^1(X; SO_n) \to H^2(X; \mathbb{Z}_2)$$

associated to the Spin_n-extension of $\mathbb{Z}_2$ via SO_n. It agrees with the ordinary definition in the special case of oriented Riemannian vector bundles.

The *third integral Stiefel-Whitney class* is

$$W_3 := \beta \circ w_2 : H^1(X; SO_n) \to H^2(X; \mathbb{Z})$$

where β is the *Bockstein homomorphism*, that is, the coboundary map associated to the extension

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0 \end{tikzcd}$$

5. K-HOMOLOGY

We are using analytic K-homology of C^*-algebras and in particular of manifolds as presented in the monograph [HR00] by Higson and Roe. A brief review of K-homology can be found in the article [PBS07].

PART II. DIRAC STRUCTURE

6. DIRAC STRUCTURE

From now on let V be an n-dimensional Riemannian vector bundle over a manifold M.

Analytic definition. [HR00] An *p-multigraded analytic Dirac structure* on V is a p-multigraded Hermitean vector bundle E with a bundle morphism

$$V \to \operatorname{End}(E)$$

such that each vector v acts by skew-adjoint, odd endomorphisms which commute with the multigrading operators and whose square is multiplication with $-||v||^2$. This action is called *Clifford multiplication.*

An *isomorphism* of p-multigraded analytic Dirac structures is an even, multigraded, V-equivariant, unitary vector bundle isomorphism.

Algebraic definition. [ST04] An *p-multigraded algebraic Dirac structure* on V is a graded Hermitean $\operatorname{Cliff}^{\mathbb{C}}(V)$–$\mathbb{C}_p$-bimodule bundle (of *-algebras). Again, the left action is called *Clifford multiplication.*

An *isomorphism* of p-multigraded algebraic Dirac structures is an even, unitary bimodule isomorphism.

Terminology. It is evident that the algebraic definition is just a rewording of the analytic one. Hence in the following we will simply call such a structure a *p-multigraded Dirac structure* unless we want to refer to the concrete picture we are using. The bundle which comprises a Dirac structure is called its *Dirac bundle.*

A *Dirac vector bundle* is a Riemannian vector bundle equipped with a Dirac structure; a vector bundle is *Dirac* if it *can* be equipped with such a structure.

Similarly, a *Dirac manifold* is a manifold whose tangent bundle is a Dirac vector bundle; a manifold is *Dirac* if its tangent bundle is Dirac.

Example 19. Any trivial vector bundle is Dirac: *Trivial* n-multigraded Dirac structures are given by fixing any orthonormal frame (E_i) and taking $E := M \times \mathbb{C}_n$ where the E_i act by left multiplication and $\mathbb{C}_n$ by right multiplication.

Example 20. The complexified Clifford algebra bundle $\operatorname{Cliff}^{\mathbb{C}}(V)$ itself is a 0-multigraded Dirac bundle by left multiplication.

Example 21 ([PBS07, 3.10])**.** Assume that V is an *oriented* Riemannian vector bundle of *even* dimension $n = 2k$. Then any local oriented orthonormal frame (E_i) over $U \subseteq M$ induces a local bundle endomorphism by *right* multiplication with $i^k E_1 \cdots E_n$. Since the transition matrix between any two such frames has determinant 1, this definition does not depend on the choice of (E_i), hence extends to a bundle endomorphism σ on all of V.

A quick calculation shows that σ is an even, self-adjoint involution which obviously commutes with the Dirac bundle structure given by *left* multiplication. It follows that the complexified Clifford algebra bundle splits into a sum of two 0-multigraded Dirac bundles

$$\operatorname{Cliff}^{\mathbb{C}}(V) =: \operatorname{Cliff}^{\mathbb{C}}_{\frac{1}{2}}(V) \overset{\perp}{\oplus} \operatorname{Cliff}^{\mathbb{C}}_{-\frac{1}{2}}(V)$$

corresponding to the ±1-eigenbundles of σ.

Periodicity.

Proposition 22. *There is a bijection of isomorphism classes of p-multigraded Dirac structures on V and isomorphism classes of $(p+2)$-multigraded Dirac structures (doubling the fiber dimension of the bundle).*

Proof. If E is a p-multigraded Dirac bundle on V then $E' := E \overset{\perp}{\oplus} E^{\mathrm{op}}$ together with the multigrading operators

$$\epsilon'_i := \epsilon_i \oplus \epsilon_i, \quad \epsilon'_{p+1} := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad \epsilon'_{p+2} := \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$$

and the diagonal action of V is a $(p+2)$-multigraded Dirac bundle on V.

Conversely, if E is a $(p+2)$-multigraded Dirac bundle with grading operators (ϵ_i). Then $e := -i\epsilon_{p+1}\epsilon_{p+2}$ is an even, self-adjoint involution and E decomposes as a direct sum of orthogonal, graded eigenbundles for e. Moreover, e commutes with $\epsilon_1, \ldots, \epsilon_p$. Hence compression to the $+1$ eigenbundle $\mathrm{Eig}_1(e)$ yields a p-multigraded Dirac bundle.

It is clear that by compressing the extension of a Dirac bundle we retrieve the same bundle. Conversely, if E is a $(p+2)$-multigraded Dirac bundle then an isomorphism between the extension of its compression and E is given by

$$\mathrm{Eig}_1(e) \overset{\perp}{\oplus} \mathrm{Eig}_1(e)^{\mathrm{op}} \to E_+ \overset{\perp}{\oplus} E_-, e \oplus e' \mapsto e - \epsilon_{p+1}(e')$$

☐

Proposition 23. *Every n-multigraded Dirac bundle of fiber dimension 2^n on an n-dimensional vector bundle V is locally trivial (in the sense of Ex. 19).*

Proof. Let us introduce operators

$$\epsilon'_i u := (-1)^{\deg u} \epsilon_i u \quad (i = 1, \ldots, n)$$

Choose also any oriented local orthonormal frame (E_i). Then we can consider any fiber as a 2^n-dimensional representation of the algebra with anticommuting generators (E_i) and (ϵ'_i) satisfying the following relations:

$$\epsilon'^2_i = 1, \quad \epsilon'^*_i = \epsilon'_i, \quad E_i^2 = -1, \quad E_i^* = -E_i$$

This algebra is isomorphic to the matrix algebra $M_{2^n}(\mathbb{C})$ which has a unique representation of fiber dimension 2^n. By applying the same argument to E locally equipped with a trivial Dirac structure, we see that both representations agree. It follows that E is locally trivial. ☐

7. DIRAC OPERATORS

Dirac operators. In the special case of Dirac manifolds, i.e. $V = TM \cong T^*M$, the action of the (co)tangent bundle on a Dirac bundle E looks suspiciously like the symbol of a first-order differential operator acting on sections of E.

A *p-multigraded Dirac operator* is an odd, symmetric first-order differential operator D acting on sections of a p-multigraded Hermitean vector bundle E over M which commutes with the multigrading operators and whose symbol σ_D satisfies

$$\sigma_D(\xi)^2 = -||\xi||^2 \qquad (\forall \xi \in T^*M)$$

Proposition 24. *The symbol of a p-multigraded Dirac operator equips the bundle it is acting on with a Clifford multiplication rendering it a p-multigraded Dirac bundle for M.*

Proposition 25. *A p-multigraded Dirac operator D defines a canonical K-homology class $[D] \in K_p(M)$.*

Proof. From the definition of a Dirac operator it follows in particular that D is elliptic. Hence we can apply the general theory of [HR00, 10.8.3]. ☐

Corollary 26. *Any p-multigraded Dirac bundle E* which has a corresponding Dirac operator *determines a canonical K-homology class $[E] \in K_p(M)$.*

Proof. Let $[E] := [D]$ for *any* Dirac operator corresponding to E. Since K-homology cannot distinguish between two operators with the same symbol (cf. [HR00, 10.9.5]), this definition does not depend on the choice of Dirac operator. $\square$

Corollary 27. *Any p-multigraded Dirac manifold M* which has a corresponding Dirac operator *defines a canonical K-homology class $[M] \in K_p(M)$.*

Proposition 28. *Multigraded periodicity of Dirac bundles is compatible with formal periodicity of K-homology. That is, if E is a p-multigraded Dirac bundle for M* which has a corresponding Dirac operator *then its $(p+2)$-multigraded extension E' also has a corresponding Dirac operator and $[E']$ is the image of $[E]$ under the periodicity isomorphism $K_p(M) \to K_{p+2}(M)$.*

Proof. By hand (sketch, assuming M closed): The direct sum operator is a Dirac operator for E'. Now, restricting a Dirac operator for E' to sections of the compressed bundle E yields a Dirac operator for E whose normalization is then used to define a Fredholm module for $[E]$ (cf. [HR00, 10.6.6]). On the other hand, compression of a Fredholm module for $[E']$ is simply restriction of the normalized Dirac operator to sections of the compressed bundle E. The claim now follows since normalization and restriction commute.

Using the Kasparov product:

Missing. TODO

$\square$

Dirac connections. It is not clear *a priori* whether every Dirac structure arises this way. In case we are given additional data we can always do it, though:

A *p-multigraded Dirac connection* for a p-multigraded Dirac bundle E on M is a $\mathbb{C}$-linear connection

$$\nabla : \Gamma^\infty(TM) \otimes \Gamma^\infty(E) \to \Gamma^\infty(E)$$

compatible with the Riemannian metric on E, the Levi-Civita connection on M and the multigrading (in the sense that ∇_X is even and commutes with the multigrading operators).

Proposition 29. *Let ∇ be a Dirac connection for a p-multigraded Dirac bundle E on M. Then the operator*

$$D : \Gamma_c^\infty(E) \xrightarrow{\nabla} \Gamma_c^\infty(T^*M \otimes E) \xrightarrow{g} \Gamma_c^\infty(TM \otimes E) \xrightarrow{\cdot} \Gamma_c^\infty(E)$$

is a p-multigraded Dirac operator for E. Furthermore, if (E_i) is a local orthonormal frame for TM then D has the local representation

$$Du = \sum_i E_i \cdot \nabla_{E_i} u$$

Proof. By compatibility of the connection with the multigrading, it is obvious that D is an odd first-order differential operator commuting with the multigrading operators.

We will now verify the local representation formula from it is immediate that the symbol of D is given by the Clifford multiplication of E.

$$u \mapsto (E_i \mapsto \nabla_{E_i} u) \cong (\sum_i E_i^* \otimes \nabla_{E_i} u) \mapsto (\sum_i E_i \otimes \nabla_{E_i} u) \mapsto \sum_i E_i \cdot \nabla_{E_i} u$$

Finally, we have to show that D is symmetric: Suppose we are given compactly supported sections u and v. Let $m \in M$ and choose an orthonormal frame (E_i) around x so that $\nabla^{\mathrm{LC}}_{E_i} E_i|_m = 0$. Then by compatibility with Levi-Civita connection and metric we have at m:

$$\langle Du_1, u_2 \rangle - \langle u_1, Du_2 \rangle|_m = \sum_i \langle E_i \cdot \nabla_{E_i} u_1, u_2 \rangle|_m - \langle u_1, E_i \cdot \nabla_{E_i} u_2 \rangle|_m$$

$$\overset{(i)}{=} \sum_i \langle \nabla_{E_i} (E_i \cdot u_1), u_2 \rangle|_m - \langle u_1, E_i \cdot \nabla_{E_i} u_2 \rangle|_m$$

$$\overset{(ii)}{=} \sum_i E_i \cdot \langle E_i \cdot u_1, u_2 \rangle = \operatorname{div}(X)$$

where X is the vector field defined by

$$\langle X, Y \rangle := \langle Y \cdot u_1, u_2 \rangle$$

Since this equality holds for any point $m \in M$ we see that

$$\langle Du_1, u_2 \rangle_2 - \langle u_1, Du_2 \rangle_2 = \int_M \operatorname{div}(X) \, dV = 0$$

□

Twistings. The following construction is used e.g. in [Pia09].

Proposition 30. *Let D be a p-multigraded Dirac operator acting on sections of a bundle E which is induced by a connection ∇^E. Let W be another Hermitean vector bundle with connection ∇^W. Then $E \otimes W$ is a p-multigraded Dirac bundle with the inherited bimodule action of E*

$$X \cdot (e \otimes w) := (X \cdot e) \otimes w, \quad (e \otimes w) \cdot x := (e \otimes X) \otimes w$$

and the tensor product connection

$$\nabla^{E \otimes W} (e \otimes w) := (\nabla^E e) \otimes w + e \otimes (\nabla^W w)$$

is a Dirac connection for $E \otimes W$.

Proposition 31. *In the situation of the previous proposition, every choice of Hermitean vector bundle W and connection ∇^W induces a canonical p-multigraded Dirac operator on $E \otimes W$ called the Dirac operator obtained from D by* twisting *with W.*

Proposition 32. *In the situation of the previous proposition, every choice of Hermitean vector bundle W induces a canonical K-homology class $[E \otimes W] \in K_p(M)$.*

Terminology in the literature. In this article we are mainly following the terminology of [HR00] so let us issue some words of caution regarding conflicting nomenclature in the literature: In the popular [LM89] the authors define a *Dirac bundle* to be an ungraded Dirac structure equipped with a Dirac connection. The same concept is called *Clifford bundle*[1] in the research notes [Roe98] (optionally with grading, i.e. 0-multigrading).

---

[1] Such a Clifford bundle should not be confused with the Clifford algebra bundle $\mathrm{Cliff}(V)$ of a Riemannian vector bundle (V, g).

8. Operations

Restriction, Cartesian product, suspension.

Proposition 33 ([HR00, 11.1.7]). *An open subset U of a Dirac manifold M inherits a canonical Dirac manifold structure by* restriction*.*

If M has an associated Dirac operator, then so does U and we have

$$\mathrm{incl}^!([M]) = [U]$$

where $\mathrm{incl}^!$ denotes the "wrong way"-homomorphism induced by the inclusion $C_0(U) \subseteq C_0(M)$.

Proposition 34 ([HR00, 11.1.8]). *The* Cartesian product *of two Dirac manifolds M_1 and M_2 is canonically a Dirac manifold by forming the graded tensor product of Dirac bundles.*

If M_1, M_2 have associated Dirac operators D_1, D_2, then $D_1 \times D_2 := D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2$ is a Dirac operator for $M_1 \times M_2$ and we have

$$[M_1 \times M_2] = [M_1] \times [M_2]$$

Proposition 35 ([HR00, 11.2.5]). *Let M be a Dirac manifold that has an associated Dirac operator and suppose that $\mathbb{R}$ is equipped with its trivial Dirac structure $\mathbb{R} \times \mathbb{C}_1$. Then*

$$s([\mathbb{R} \times M]) = [M]$$

where $s : K_{*+1}(\mathbb{R} \times M) \to K_*(M)$ denotes the suspension isomorphism *of K-homology.*

Part III. Spin^c structure

9. Spin^c structure

We will now give several definitions of Spin^c structure which later turn out to be equivalent.

Assume that $n \geqslant 3$.

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Geometric definition. [LM89] A *geometric Spin^c structure* on V consists of the following data:

- an orientation on V
- a principal Spin^c_n-bundle $P_{\mathrm{Spin}^c_n}(V)$ over M
- a principal U_1-bundle $P_{U_1}(V)$ over M
- a vertical principal bundle morphism

$$\begin{tikzcd}
\mathrm{Spin}^c_n \arrow[rr, "\xi_0"] \arrow[d] & & SO_n \times U(1) \arrow[d] \\
P_{\mathrm{Spin}^c_n}(V) \arrow[rr, "\xi"] \arrow[dr] & & P_{SO_n}(V) \times P_{U_1}(V) \arrow[dl] \\
& M &
\end{tikzcd}$$

Two geometric Spin^c structures are *isomorphic* if they have the same orientation and their principal Spin^c_n-bundles are equivalent.

Cohomologic definition. [BD82] A *cohomological Spin^c structure* on V is given by an orientation and a cohomology class in $H^2(P_{SO_n}(V)|_U;\mathbb{Z})$ for each connected component $U \subseteq M$ which restricts to the generator of the cohomology group $H^2(SO_n;\mathbb{Z}) \cong \mathbb{Z}_2$ of any fiber.

Analytic definition. [HR00] An *analytic Spin^c structure* on V is an n-multigraded analytic Dirac structure which is locally trivial (in the sense of Ex. 19). In this case, the Dirac bundle is called the *full spinor bundle* $S_{\text{full}}(V)$.

Two analytic Spin^c structures are *isomorphic* if there are isomorphic in the sense of analytic Dirac bundles.

Algebraic definition. [ST04] An *algebraic Spin^c structure* on V is an

TODO

irreducible

n-multigraded algebraic Dirac structure.

Two algebraic Spin^c structures are *isomorphic* if there are isomorphic in the sense of algebraic Dirac bundles.

Terminology. We will later prove that these definitions are equivalent, i.e. different pictures of the same data. In the following we will use the term *Spin^c structure* if we do not want to refer to a specific picture.

A *Spin^c-vector bundle* is a Riemannian vector bundle equipped with a Spin^c structure; a vector bundle is *Spin^c* if it *can* be equipped with such a structure.

Similarly, a *Spin^c-manifold* is a manifold whose tangent bundle is a Spin^c-vector bundle, and a manifold is *Spin^c* if its tangent bundle is Spin^c.

10. EXISTENCE

Proposition 36. *An oriented Riemannian vector bundle V is geometrically Spin^c if and only if its second Stiefel-Whitney class is the* mod 2 *reduction of an integral class in $H^2(M;\mathbb{Z})$.*

Proof. We have a commutative diagram of extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d, "="'] & \mathrm{Spin}_n \arrow[r] \arrow[d, "\subseteq"'] & SO_n \arrow[r] \arrow[d, "\subseteq"'] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathrm{Spin}^c_n \arrow[r] & SO_n \times U_1 \arrow[r] & 0
\end{tikzcd}$$

The long exact sequence corresponding to the latter extension is

$$\cdots \longrightarrow H^1(M;\mathrm{Spin}^c_n) \xrightarrow{\xi^*} H^1(M;SO_n) \oplus H^1(M;U_1) \xrightarrow{\partial} H^2(M;\mathbb{Z}_2) \longrightarrow \cdots$$

Clearly, V is geometrically Spin^c if and only if its principal SO_n-bundle $P_{SO_n}(V)$ is the first component of some element in the range of ξ^*, hence by exactness in the kernel of ∂. This latter map decomposes as

$$\partial = w_2 \oplus (\pi^* \circ c_1)$$

where π is the projection $\mathbb{Z} \to \mathbb{Z}_2$ (this is shown by concretely calculating the boundary maps; philosophically the w_2-part comes from the above diagram and the other part is due to the correspondence $\lambda \mapsto \lambda^2 \leftrightsquigarrow n \mapsto 2n$ under taking the logarithm). Thus, V is geometrically Spin^c if and only if

$$w_2(P_{SO_n}(V)) = \pi^*(u)$$

for some $u \in H^2(M;\mathbb{Z}) \xleftarrow[c_1]{\cong} H^1(M;U_1)$. ☐

Proposition 37 ([BD82]). *An oriented Riemannian vector bundle V is cohomologically Spin^c if and only if its second Stiefel-Whitney class is the* mod 2 *reduction of an integral class in $H^2(M;\mathbb{Z})$.*

We summarize the preceding without reference to any specific definition of Spin^c structure.

Corollary 38. *An oriented Riemannian vector bundle V is Spin^c if and only if its second Stiefel-Whitney class is the* mod 2 *reduction of an integral class in $H^2(M;\mathbb{Z})$, i.e. if and only if its third integral Stiefel-Whitney class vanishes.*

Proof. Apply $\mathrm{ran}(\pi^*) = \ker(\beta)$ to the preceding. ☐

11. Classification

Proposition 39. *Let M be connected. Then $\mathrm{Prin}_{U_1}(M)$ operates freely and transitively on the set of isomorphism classes of geometric Spin^c structures by* twisting, *i.e. multiplication of cocycles.*

Proof. Free: Suppose P is a Spin^{c}structure with cocycles

$$[g_{i,j}, h_{i,j}] : U_{i,j} \to \mathrm{Spin}^c_n = \mathrm{Spin}_n \times_{\mathbb{Z}_2} U_1$$

and L is a principal U_1-bundle given by cocycles $(\lambda_{i,j})$ with respect to some good cover (U_i) such that the isomorphism class of P is invariant under twisting with L, i.e.

$$[a_{i,j} g_{i,j} a_{i,j}^{-1}, h_{i,j}] = [g_{i,j}, h_{i,j}\lambda_{i,j}]$$

(since U_1 is Abelian). Since the Spin$_n$ parts agree after postcomposition with xi_0 we can assume without loss of generality that they agree (otherwise multiply one of them with -1). It follows that $\lambda_{i,j} \equiv \pm 1$. If it were -1 then the cocycle condition would be violated; hence all $\lambda_{i,j} \equiv 1$.

Transitive: Suppose P^k $(k = 1, 2)$ are Spin^{c}structures given by cocycles

$$[g^k_{i,j}, h^k_{i,j}] : U_{i,j} \to \mathrm{Spin}^c_n = \mathrm{Spin}_n \times_{\mathbb{Z}_2} U_1$$

with respect to some good cover (U_i). Again, since $\xi_0 \circ g^1_{i,j} = \xi_0 \circ g^2_{i,j}$ we can assume without loss of generality that $g^1_{i,j} = g^2_{i,j}$ (otherwise multiply one of the cocycles by -1). But then $\lambda_{i,j} := h^2_{i,j}/h^1_{i,j}$ defines a principal U_1-bundle which clearly twists P^1 to P^2. ☐

Proposition 40. *Let M be connected. Then $\mathrm{Prin}_{U_1}(M)$ operates freely and transitively on the set of cohomologic Spin^c structures considered as a subset of $\mathrm{Prin}_{U_1}(P_{SO_n}(V))$ by multiplication of (pulled back) cocycles.*

Proof. The long exact cohomology sequence for the fibration $P_{SO_n}(V)$ is

$$\cdots \longrightarrow H^1(SO_n;\mathbb{Z}) \longrightarrow H^2(M;\mathbb{Z}) \xrightarrow{\pi^*} H^2(P_{SO_n}(V);\mathbb{Z}) \xrightarrow{\text{incl}^*} H^2(SO_n;\mathbb{Z}) \longrightarrow \cdots$$

Since $H^1(SO_n;\mathbb{Z}) = 0$ and $H^2(SO_n;\mathbb{Z}) = \mathbb{Z}_2$, the set of Spin^c structures corresponds to the coset of $H^2(P_{SO_n}(V);\mathbb{Z})$ which maps to the generator of $H^2(SO_n;\mathbb{Z})$. The cohomology group $H^2(M;\mathbb{Z})$ acts freely and transitively on this set via π^*. But under the natural isomorphism $H^2(\cdot;\mathbb{Z}) \cong H^1(\cdot;U_1) \cong \text{Prin}_{U_1}(\cdot)$ this is just the pullback to $P_{SO_n}(V)$ and multiplication in that group is given by pointwise multiplication of cocycles. ☐

Again, we summarize the preceding without any reference to the definition of Spin^c structure we are using.

Corollary 41. *The set of Spin^c structures on a Spin^c-vector bundle (manifold) is classified by* $H^2(M;\mathbb{Z}) \cong H^1(M;U_1) \cong \text{Prin}_{U_1}(M) \cong \text{Vect}_1^{\mathbb{C}}(M)$.

12. EQUIVALENCE OF DEFINITIONS

Proposition 42. *The concepts of geometric and cohomologic Spin^c structures are equivalent.*

More precisely, V is geometrically Spin^c if and only if it is cohomologically Spin^c, and for any fixed orientation on V the map

$$\begin{cases} \{\textit{geometric Spin}^c \textit{ structures}\}/\cong \longrightarrow \{\textit{cohomologic Spin}^c \textit{ structures}\} \\ [P_{Spin_n^c}] \mapsto [P_{Spin_n^c}] \end{cases}$$

induced by considering principal Spin$_n^c$-bundles over M as principal U_1-bundles over $P_{SO_n}(V)$ is a bijection.

Proof. The first statement follows from combining Prop. 36 and 37. Now consider the given map. It is well-defined because ξ_0 is not trivial on the fibers, and since it intertwines the transitive and free twisting actions of Prop. 39 and 40 it is a bijection. ☐

Proposition 43. *The concepts of geometric and analytic Spin^c structures are equivalent.*

Proof. The following constructions are inverse to each another (up to isomorphism).

($\Rightarrow$) Assume that we have a geometric Spin^c structure on V. Then we can combine the standard isomorphism

$$\text{Cliff}^{\mathbb{C}}(V) \cong P_{SO_n}(V) \times_{SO_n} \mathbb{C}_n$$

(standard action of SO_n; see Prop. 18) with the isomorphism

$$\begin{cases} P_{\text{Spin}_n^c}(V) \times_{\text{Spin}_n^c} \mathbb{C}_n \to P_{SO_n}(V) \times_{SO_n} \mathbb{C}_n \\ [p,v] \mapsto [(\text{proj}_1 \circ \xi_0)(p), v] \end{cases}$$

(adjoint action of Spin$_n^c$; it descends to the standard action of SO_n on $\mathbb{C}_n$ via $\text{proj}_1 \circ \xi_0$; the isomorphism is well-defined precisely because ξ agrees fiberwise with ξ_0).

Define the graded vector bundle

$$E := P_{\text{Spin}_n^c}(V) \times_{\text{Spin}_n^c} \mathbb{C}_n$$

(left multiplication action of Spin_n^c) which is naturally a right $\mathbb{C}_n$-module with right multiplication. It is also a left $\mathrm{Cliff}^{\mathbb{C}}(V)$-module with left multiplication

$$[(p,x)]\cdot[(p,y)] := [(p,xy)]$$

which is well-defined since

$$[(pg^{-1},\mathrm{ad}(g)(x)gy)] = [(pg^{-1},gxg^{-1}gy)] = [(pg^{-1},gxy)] = [(p,xy)]$$

Left and right multiplication commute, hence they combine to yield a bimodule structure on E. It is also clear that both actions respect the grading since this is so on the level of $\mathbb{C}_n$. We can make this a bimodule of $*$-algebras by choosing a Hermitean inner product on E which is invariant under left and right multiplication of unit vectors (by averaging over the action of the respective Clifford groups, see [LM89, p. 37]).

Finally, if (E_i) is a local orthonormal frame over $U \subseteq M$ then we have trivializations

$$\begin{aligned}\mathrm{Cliff}^{\mathbb{C}}(V)|_U &\cong U\times\mathbb{C}_n\\ E|_U &\cong U\times\mathbb{C}_n\end{aligned}$$

where the left action of $E_i\otimes 1$ is left multiplication with e_i and the right action is just right multiplication. This shows local triviality.

($\Leftarrow$): Assume we are given a locally trivial n-multigraded Dirac structure E.

We will first show how to recover an orientation: Suppose that E is trivialized with respect to two orthonormal frames (E_i) and (F_j) over an open subset $U\subseteq M$, and consider the automorphism

$$x\mapsto(-1)^{n+(n-1)(\deg x)}F_n\cdots F_1\cdot x\cdot e_1\cdots e_n \in \mathrm{End}(E|_U)$$

In the (E_i)-picture, this is multiplication by the determinant of the transition matrix between the orthonormal frames. But in the (F_j)-picture, this is simply the identity. Hence both frames have the same orientation. In particular, this yields an orientation on V: a basis is oriented if it has the same orientation as any trivializing orthonormal frame.

Define the space

$$\begin{aligned}F := \{&(m,f,g) : V_m \xrightarrow{f} \mathbb{R}^n \text{ orientation-preserving isometry},\\ &E_m \xrightarrow{g} \mathbb{C}_n \text{ even isometry},\\ &g(v\cdot e) = f(v)\cdot g(e) \text{ and } g(e\cdot w) = g(e)\cdot w\ (\forall v\in V_m, e\in E_m, w\in\mathbb{C})\}\end{aligned}$$

It is a fiber bundle over M with the appropriate topology, and there is a right Spin_n^c-action

$$(m,f,g)\cdot x := (m,\xi_0(x^{-1})\circ f,\lambda(x^{-1})\circ g)$$

which is free (λ is right multiplication). But this action is also transitive on each fiber! Indeed, suppose we are given $(m,f,g),(m,f',g')\in F$. Then $f'\circ f^{-1}$ is given by an element $A\in SO_n$, and the "transition map" $\varphi := g'\circ g^{-1}$ satisfies

$$\begin{aligned}\varphi(e_i) &= \varphi(e_i\cdot 1) = (Ae_i)\varphi(1)\\ \varphi(x) &= \varphi(1x) = \varphi(1)x\\ \Rightarrow \varphi(1)e_i\varphi(1)^{-1} &= Ae_i\end{aligned}$$

Thus if we can show that $\varphi(1)\in\mathrm{Spin}_n^c$ it follows that $\varphi(1)$ is a lift of A and its action sends (m,f',g') to (m,f,g). Since $\varphi(1)$ is an even element of

norm 1

TODO

this is indeed the case. □

In particular, we see from the proof that an analytic Spin^c structure on a vector bundle V induces a canonical orientation on V. We will always consider V being equipped with this orientation.

We could have also proved this last proposition in the same fashion as the previous proposition (cf. [HR00, top of p. 318]).

Proposition 44. *The concepts of analytic and algebraic Spin^c structures are equivalent.*

TODO

Proof. how can we have both multigrading and irreducibility!?

□

Hence all the definitions we gave in Sec. 9 are indeed equivalent.

13. Spinor bundles

Spinor bundles. Let V be a Spin^c-vector bundle of dimension n. If S is a graded $*$-representation of $\mathbb{C}_n$ then the associated bundle

$$S(V) := P_{\mathrm{Spin}^c_n}(V) \times_{\mathrm{Spin}^c_n} S$$

is called a *spinor bundle*. If S also has a right graded $\mathbb{C}_p$-action commuting with the left action then $S(V)$ is p-multigraded. We designate the spinor bundles of a Spin^c-manifold M by $S(M) := S(TM)$.

Example 45. The *full spinor bundle $S_{full}(V)$* is indeed the n-multigraded spinor bundle induced by the full spinor representation of $\mathbb{C}_n$ on itself by left multiplication.

Proposition 46. *Every spinor bundle $S(V)$ is a $\mathrm{Cliff}^{\mathbb{C}}(V)$-module bundles via*

$$\mathrm{Cliff}^{\mathbb{C}}(V) \cong P_{\mathit{Spin}^c_n}(V) \times_{\mathit{Spin}^c_n} \mathbb{C}_n \circlearrowleft S(V) = P_{\mathit{Spin}^c_n}(V) \times_{\mathit{Spin}^c_n} S,$$
$$[p, x] \cdot [p, v] := [p, x \cdot v]$$

In particular, every p-multigraded spinor bundle is a p-multigraded Dirac bundle.

Proof. We have seen both the isomorphism and the action in the proof of Prop. 43. □

The spinor bundle associated to an graded irreducible representation S is called *fundamental*. Since every such representation restricts to the same representation on Spin^c_n, geometrically there is only a single *fundamental spinor bundle $S_{fund}(V)$*.

Proposition 47. *Multigraded periodicity from Sec. 6 restricts to multigraded spinor bundles.*

If V is a Spin^c-vector bundle of dimension n then using the periodicity bijection we can reduce its full spinor bundle $S_{\mathrm{full}}(V)$ in $\lfloor n/2 \rfloor$ steps to an (n mod 2)-multigraded spinor bundle, called the *reduced spinor bundle* $S_{\mathrm{red}}(V)$ of V. Its fibers have dimension $2^{\lceil n/2 \rceil}$, hence consist of graded irreducible modules (by Cor. 12). In view of Prop. 23 we have proven the following proposition.

Proposition 48. *Reduced spinor bundles are fundamental.*

Furthermore, if $\dim V$ *is even, then there is a one-to-one correspondence between its* 0*-multigraded fundamental spinor bundles and full spinor bundles, and if* $\dim V$ *is odd, then there is a one-to-one correspondence between its* 1*-multigraded fundamental spinor bundles and full spinor bundles.*

Relations with the complexified Clifford algebra bundle. The complexified Clifford algebra bundle $\mathrm{Cliff}^{\mathbb{C}}(V)$ of a Spin^c vector bundle V is *not* a 0-multigraded spinor bundle but only a 0-multigraded Dirac bundle: Although we have seen that

$$\mathrm{Cliff}^{\mathbb{C}}(V) \cong P_{\mathrm{Spin}^c_n}(V) \times_{\mathrm{Spin}^c_n} \mathbb{C}_n$$

with the *adjoint* representation, the left action of $\mathrm{Cliff}^{\mathbb{C}}(V)$ is given by left multiplication (see Ex. 20) and not by the adjoint representation (which would not be graded anyway).

Still there are some useful relations between $\mathrm{Cliff}^{\mathbb{C}}(V)$ and certain spinor bundles of V, at least in the *even-dimensional* case. The following proposition is the obvious analogue of Prop. 15.

Proposition 49. *If* V *is a Spin*^{c}*vector bundle of even dimension then*

$$\mathrm{Cliff}^{\mathbb{C}}(V) \cong \mathrm{End}(S_{red}(V)) \cong S_{red}(V) \hat{\otimes} S^*_{red}(V)$$

of 0*-multigraded Dirac bundles.*

Proposition 50. *The grading operator* Ω *of the reduced spinor bundle of a Spin*^{c}*vector bundle* V *of even dimension* $n = 2k$ *is given by left multiplication with*

$$i^k E_1 \cdots E_n$$

where (E_i) *are any local oriented orthonormal frames.*

Proof. It is clear from Ex. 13 the grading operator is one of $\pm\Omega$. Note that $-\Omega$ is the operator we get by using *wrongly* oriented orthonormal frames In the following we will show that the grading operator indeed arises from propertly oriented orthonormal frames.

We have seen in the proof of Prop. 43 that the bundle morphism

$$x \mapsto i^k E_1 \cdots E_n \cdot x \cdot \omega \in \mathrm{End}(S_{\mathrm{full}}(V))$$

is the grading morphism for the full spinor bundle.

Now let Δ_{2k} be the standard graded irreducible representation of $\mathbb{C}_{2k}$ from Ex. 13. Prop. 23 shows that

$$S_{\mathrm{red}}(V) \hat{\otimes} \Delta_{2k} \cong S_{\mathrm{full}}(V)$$

and we conclude that the grading morphism of $S_{\mathrm{red}}(V) \hat{\otimes} \Delta_{2k}$ is given by

$$s \otimes v \mapsto \Omega s \otimes s \cdot \omega$$

On the other hand, this grading morphism has to be the tensor product of the grading morphisms of the factors. But the grading morphism of Δ_{2k} is given by right multiplication with ω. Hence Ω is indeed the grading morphism of $S_{\mathrm{red}}(V)$. ☐

Proposition 51 ([PBS07, 4.12])**.** *If* V *is a Spin*^{c}*vector bundle of even dimension* $n = 2k$ *then*

$$\mathrm{Cliff}^{\mathbb{C}}_{\frac{1}{2}}(V) \cong \mathrm{Hom}(S_{red,+}(V), S_{red}(V)) \cong S_{red}(V) \hat{\otimes} S^*_{red,+}(V)$$

of 0*-multigraded Dirac bundles (*$S_{red,+}(V)$ *being trivially graded).*

Proof. Recall from Ex. 21 that $\mathrm{Cliff}^{\mathbb{C}}_{\frac{1}{2}}(V)$ consists precisely of those elements which are invariant under σ. Together with Prop. 49 we see that their actions are precisely those endomorphisms which are invariant under Ω. It follows that

$$\begin{aligned} &\{\phi \in \mathrm{End}(S_{\mathrm{red}}(V)) : \phi(\Omega s) = \phi(s)\ (\forall s)\} \\ =&\{\phi \in \mathrm{End}(S_{\mathrm{red}}(V)) : \phi|_{S_{\mathrm{red},-}(V)} = 0\} \\ =&\,\mathrm{Hom}(S_{\mathrm{red},+}(V), S_{\mathrm{red}}(V)) \end{aligned}$$

□

Dirac operators. Let us consider the case of Spin^c-manifolds.

Proposition 52. *Let $S(M)$ be a p-multigraded spinor bundle on a Spin^c-manifold M. Then any choice of connection on $P_{U_1}(M)$ yields a p-multigraded Dirac connection and hence a p-multigraded Dirac operator for $S(M)$.*

In particular, every p-multigraded spinor bundle determines a canonical K-homology class $[S(M)] \in K_p(M)$.

Proof. The Levi-Civita connection on M induces a connection on the orthonormal frame bundle $P_{SO_n}(M)$. If we also choose a connection on the principal U_1-bundle $P_{U_1}(M)$ then we get a connection on the product bundle $P_{SO_n}(M) \times P_{U_1}(M)$ which we can lift onto the two-fold covering $P_{\mathrm{Spin}^c_n}(M)$. It extends trivially to $P_{\mathrm{Spin}^c_n}(M) \times SO_n$. Since the left action restricted to Spin^c_n is unitary, the connection descends to $P_{\mathrm{Spin}^c_n}(M) \times_{\mathrm{Spin}^c_n} SO_n \cong P_{SO_n}(S(M))$, i.e. to a connection ∇ on $S(M)$ compatible with the metric. One can verify that ∇ is compatible with the Levi-Civita connection [LM89, Prop. 4.11]. Also, it is obvious that its holonomy is given by left multiplication with certain elements of Spin^c_n [LM89, p. 139]. It follows from that latter fact that ∇ is compatible with the multigrading. Putting these arguments together we see ∇ is a multigraded Dirac connection for $S(M)$. Now we can apply the theory of Sec. 7. □

Note in particular that these Dirac operators can be twisted (in the sense of Sec. 7).

Specializing to the full spinor bundle $S_{\mathrm{full}}(M)$ of M we get the following important result:

Corollary 53. *Every n-dimensional Spin^c-manifold M has a n-multigraded Dirac operator D_M (canonical up to choice of connection on $P_{U_1}(M)$) and hence determines a canonical K-homology class $[M] \in K_n(M)$, the* fundamental *K-homology class of M.*

Corollary 54. *Multigraded periodicity of spinor bundles is compatible with formal periodicity of K-homology. That is, if $S(M)$ is a p-multigraded spinor bundle and $S'(M)$ its $(p+2)$-multigraded extension, then $[S'(M)]$ is the image of $[S(M)]$ under the periodicity isomorphism $K_p(M) \to K_{p+2}(M)$.*

Summary.

$$\mathrm{Spin}^c \text{ structure} \xrightarrow[\text{bundle}]{\text{spinor}} \text{Dirac structure} \xrightarrow[\text{connection}]{\text{Dirac}} \text{Dirac operator}$$

14. Operations

In this section we will present various ways to create new Spin^c structures from existing ones. It is instructive to see that different operations are better described in some pictures than others. We will also analyze what happens with the associated K-homology classes under these operations.

Restriction.

Proposition 55. *Any open subset U of a Spin^c-manifold M carries a canonical Spin^c-manifold structure by* restriction.

Furthermore,

$$\mathrm{incl}^!([M]) = [U]$$

where $\mathrm{incl}^!$ *denotes the "wrong way"-homomorphism induced by the inclusion $C_0(U) \subseteq C_0(M)$.*

Whitney sum, Cartesian product, submanifold, boundary.

Proposition 56. *Given Riemannian vector bundles $V \cong V' \overset{\perp}{\oplus} V''$, a choice of Spin^c structure on two of them determines a canonical Spin^c structure on the third.*

Proof. The corresponding statement for orientations is clear, thus assume that we have already oriented the third bundle such that V carries the sum orientation. Now from

$$W_3(V) = W_3(V') + W_3(V'')$$

(which follows from the analogue statement for w_2 which in turn follows from the vanishing of w_1 for orientable vector bundles) we see that if any two of the bundles is Spin^c then so is the third. Without loss of generality assume that M is connected.

Fix arbitrary cohomologic Spin^c structures $a' \in H^2(P_{SO_{n'}}(V'); \mathbb{Z})$ and $a'' \in H^2(P_{SO_{n''}}(V''); \mathbb{Z})$ on V' and V''. The inclusion $P_{SO_{n'}}(V') \times P_{SO_{n''}}(V'') \subseteq P_{SO_{n'+n''}}(V' \overset{\perp}{\oplus} V'')$ yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^2(M \times M; \mathbb{Z}) \arrow[r, "\pi^*"] \arrow[d, "\mathrm{id}"', "\cong"] & H^2(P_{SO_{n'+n''}}(V' \times V''); \mathbb{Z}) \arrow[r, "\mathrm{incl}^*"] \arrow[d, "\mathrm{incl}^*"', "\cong"] & H^2(SO_{n'+n''}; \mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{incl}^*"', "\cong"] & H^3(M \times M; \arrow[d, "\mathrm{id}"', "\cong"] \\
0 \arrow[r] & H^2(M \times M; \mathbb{Z}) \arrow[r, "\pi^*"] & H^2(P_{SO_{n'}}(V') \times P_{SO_{n''}}(V''); \mathbb{Z}) \arrow[r, "\mathrm{incl}^*"] & H^2(SO_{n'} \times SO_{n''}; \mathbb{Z}) \arrow[r] & H^3(M \times M;
\end{tikzcd}$$

(cf. proof of Prop. 40). Thus there is a single $b \in H^2(P_{SO_{n'+n''}}(V' \times V''); \mathbb{Z})$ corresponding to $a' \times 1 + 1 \times a'' \in H^2(P_{SO_{n'}}(V') \times P_{SO_{n''}}(V''); \mathbb{Z})$, and we can pull back this class via the diagonal map

$$\Delta : P_{SO_{n'+n''}}(V) \to P_{SO_{n'+n''}}(V' \times V'')$$

to a cohomologic Spin^c structure $a := \Delta^* b \in H^2(P_{SO_{n'+n''}}(V)$.

Now we can write any Spin^c structure on V' in the form $a' + \pi_{V'}^*(c')$, likewise any Spin^c structure on V'' in the form $a'' + \pi_{V''}^*(c'')$ and any Spin^c structure on V in the form $a + \pi_V^*(c' + c'')$. It is obvious that any choice of two of c, c' and $c' + c''$ determines the third. □

Corollary 57. *The* Cartesian product *of two $Spin^c$-manifolds M_1 and M_2 is canonically a $Spin^c$-manifold.*

In the analytic picture, $S_{full}(M_1)\hat{\otimes}S_{full}(M_2)$ is a full spinor bundle over $M_1 \times M_2$.

Furthermore, if D_1, D_2 are Dirac operators for M_1, M_2, then $D_1 \times D_2 := D_1\hat{\otimes}1 + 1\hat{\otimes}D_2$ is a Dirac operator for $M_1 \times M_2$, and we have

$$[M_1 \times M_2] = [M_1] \times [M_2]$$

Corollary 58. *Any* submanifold *with a $Spin^c$ structure on its normal bundle is canonically a $Spin^c$-manifold.*

Corollary 59 ([HR00, 11.2.15])**.** *The* boundary *of a $Spin^c$-manifold is canonically a $Spin^c$-manifold.*

In the geometric picture, by completing each frame with the normal vector we can consider

$$P_{SO_{n-1}}(\partial M) \times P_{U_1}(M)|_{\partial M} \subseteq P_{SO_n}(M) \times P_{U_1}(M)|_{\partial M}$$

and restrict $Spin^c$ structure $P_{Spin^c_n}(M)|_{\partial M}$ being a two-fold covering of the latter to the former bundle.

In the analytic picture, consider the automorphism

$$u \mapsto (-1)^{\deg u}\nu \cdot u \cdot e_1$$

of $S_{full}(M)$ (ν being the inward pointing unit normal vector). It is even, self-adjoint, has square one and commutes with Clifford multiplication with orthogonal vector and the right action of the other generators of $\mathbb{C}_n$. It follows that the (-1)-eigenbundle determines a $Spin^c$ structure of ∂M.

Furthermore, the boundary map of K-homology is compatible with taking the boundary of a $Spin^c$-manifold:

$$\partial[M] = [\partial M]$$

Pullback.

Proposition 60. *The pullback of a $Spin^c$-vector bundle is canonically a $Spin^c$-vector bundle.*

Proof. Let $f : M \to N$ be a smooth function and let V be a Spin^c-vector bundle over N. If we equip the pullback f^*V with the canonical pullback orientation and metric then the bundle map is an orientation-preserving isometry and we have a commutative diagram

$$\begin{tikzcd}
 & P_{\mathrm{Spin}^c_n}(V) \arrow[d, "2"] \\
P_{SO_n}(f^*V) \times f^*P_{U_1}(V) \arrow[r] \arrow[d] & P_{SO_n}(V) \times P_{U_1}(V) \arrow[d] \\
M \arrow[r] & N
\end{tikzcd}$$

We can now pull back the two-fold cover and find a Spin^c structure for f^*V. □

Lifting.

Proposition 61. *The total space of a Spin^c-vector bundle E over a Spin^c-manifold M has a canonical Spin^c-manifold structure (unique up to concordance).*

Proof. We have the following short exact sequence of vector bundles over E:

$$\begin{tikzcd} 0 \arrow[r] & \pi_E^*(E) \arrow[r] & T(E) \arrow[r] & \pi_E^*(TM) \arrow[r] & 0 \end{tikzcd}$$

Choose a Riemannian metric on E such that

$$T(E) \cong \pi_E^*(E) \overset{\perp}{\oplus} \pi_E^*(TM)$$

Thus we can equip $T(E)$ with the canonical direct sum-Spin^c structure.

It is also clear by convexity of the space of Riemannian metrics that different choices of Riemannian metric lead to concordant Spin^c structures. ☐

Opposite. The *opposite* of a Spin^c structure on a vector bundle V is defined in the analytic picture by negating the first multigrading operator ϵ_1.

We denote by $-V$ the same vector bundle equipped with the opposite Spin^c structure. Similarly, if M is a Spin^c-manifold then we denote by $-M$ the same manifold equipped with the opposite Spin^c structure on its tangent bundle.

Proposition 62. *The opposite Spin^c structure carries the opposite orientation.*

Proof. Considering the way we defined the canonical orientation of an analytic Spin^c structure (see proof of Prop. 43) this is obvious. ☐

Proposition 63. *The opposite Spin^c structure has the following representation in the other pictures:*

In the geometric picture, denote by $P_{SO_n}(-V)$ the oppositely oriented principal SO_n-bundle of V. Using the bundle isomorphism $P_{SO_n}(-V) \times P_{U_1}(V) \to P_{SO_n}(V) \times P_{U_1}(V)$ (induced by multiplying with -1 the first vector of an oriented frame) we can pull back the principal Spin^c_n-bundle (since it is a two-fold covering of the latter).

In the cohomologic picture,

see [BD82, page after (8.5)] TODO

.

In the algebraic picture, passing to the opposite Spin^c structure simply amounts to negating the right action of e_1.

15. Equivalence relations

Concordance. Two Spin^c-manifolds M_1, M_2 *over the same smooth manifold M* are *concordant* if $\mathbb{R} \times M$ can be equipped with a Riemannian metric and Spin^c structure which over some nonempty open intervals I_i restrict to the canonical product metric and Spin^c structure of $I_i \times M_i$ $(i = 1, 2)$.

Proposition 64. *All operations defined in the last section are well-defined for concordance classes.*

Proposition 65. *Concordant Spin^c-manifolds M_1, M_2 have the same K-homology class:*

$$[M_1] = [M_2]$$

Proof. Any orientation-preserving homeomorphism $f : I_i \to \mathbb{R}$ induces a homotopy inverse $(f \times \mathrm{id})^*$ for the "inclusion" $C_0(I_i) \hookrightarrow C_0(\mathbb{R})$. Hence the "wrong-way" homomorphism $\mathrm{incl}^!$ is invertible and we have

$$[M_i] = s([\mathbb{R} \times M_i]) = s({\mathrm{incl}_i^!}^{-1}([I_i \times M_i])) = s({\mathrm{incl}_i^!}^{-1}([I_i \times M])) = s([\mathbb{R} \times M])$$

□

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