Derivation of the Navier-Stokes equation*

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1 Coordinates

We consider a region $\Omega_0 \subseteq \mathbb{R}^3$ which will change in shape over time (since this is what fluids tend to do). Let

$$\Phi : \Omega_0 \times [0, T] \to \mathbb{R}^3$$

be the function which assigns to each point x_0 at time 0 its position $x = \Phi(x_0, t)$ at time t. The following picture illustrates the situation at hand:

x_0 $x = \Phi(x_0, t)$

Ω_0

Ω_t

Note that we can as well think of Ω_0 as the set of particles. It is thus natural to introduce *Lagrangian coordinates* (x_0, t) which consist of a particle and a point in time. This is a common choice in solid-state physics.

In fluid mechanics however one typically uses *Eulerian coordinates* (x, t) consisting of a position $x \in \Omega_t$ at a particular time t, thus focusing on individual points in space. Typical quantities expressed in this way are (by a slight misuse of notation):

1. velocity $u : \Omega_t \times [0, T] \to \mathbb{R}^3$
2. pressure $p : \Omega_t \times [0, T] \to \mathbb{R}$
3. density $\varrho : \Omega_t \times [0, T] \to \mathbb{R}$

In the following we will only discuss the case $\Omega_t = \Omega = const$.

2 Convective derivative and transport theorem

Fix any particle $x_0 \in \Omega_0$. Its *trajectory* is given by the function

$$\varphi : [0, T] \to \mathbb{R}^3, t \mapsto \Phi(x_0, t)$$

Now, given any C^1 function $h : \Omega \times [0, T] \to \mathbb{R}$, we can study its time evolution along the trajectory of x_0:

$$\tilde{h}(t) := h(\varphi(t), t)$$

By the definition of the velocity field u and using Eulerian

*After a seminar talk by HDoz. Dr. Peer Kunstmann coordinates, we find that

$$\frac{d}{dt}\tilde{h}(t) = \left(\frac{\partial}{\partial t}h\right)(\varphi(t),t) + \varphi'(t)\cdot(\nabla h)(\varphi(t),t)$$
$$= \left(\frac{\partial}{\partial t}h\right)(x,t) + u(x,t)\cdot(\nabla h)(x,t)$$

This derivative, which is given by the differential operator

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + u\cdot\nabla$$

is called the *convective derivative* of h. It thus describes the quantity's rate of change along a certain trajectory.

1 Example. By taking $h = u_j$ to be the j-th component of the velocity field, we find that the *acceleration* along a trajectory is given by

$$\frac{Du}{Dt} = \frac{\partial}{\partial t}u + \underbrace{(u\cdot\nabla u_j)_j}_{=:u\cdot\nabla u}$$

The following theorem characterizes the rate of change of volume integrals of a given quantity (which in a sense is a generalization of the convective derivative to "trajectories" of entire volumes instead of a single particle).

2 Theorem (Transport theorem)**.** *Let* $V_0 \subseteq \Omega_0$ *be a region and* $V_t := \Phi(V_0,t) \subseteq \Omega_t$*. Then we have:*

$$\frac{d}{dt}\int_{V_t} h(x,t)\cdot dx = \int_{V_t}\left(\frac{\partial h}{\partial t} + \operatorname{div}(hu)\right)(x,t)\cdot dx$$

3 Conservation of mass

The *mass* of a volume V_0 is conserved with respect to time:

$$\int_{V_t}\varrho(x,t)\cdot dx = const$$

Thus, by the transport theorem we get

$$0 = \int_{V_t}\left(\frac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho u)\right)(x,t)\cdot dx$$

for *any* volume V_0. Thus

$$0 = \frac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho u)$$

A fluid is called *homogeneous* if density does not vary over space. It is called *incompressible* if density does not vary over time.

Thus for incompressible, homogeneous fluids we have $\varrho(x,t) =: \varrho_0 = const$, and from the conservation of mass it follows that u is divergence free (or *solenoidal*):

$$\operatorname{div}(u) = 0$$

4 Momentum and forces

The *momentum* v of a volume V_0 is given by

$$v(t) := \int_{V_t}\varrho(x,t)u(x,t)\cdot dx$$

By Newton, the change of momentum is given by the sum of forces acting on the volume, and together with the transport theorem we have

$$\text{"sum of forces"} = \frac{d}{dt}v(t) = \int_{V_t}\frac{\partial(\varrho u)}{\partial t} + (\operatorname{div}(\varrho u_j u))_j\cdot dx \tag{1}$$

In the given physical situation, there are the following kinds of forces:

1. *Volume forces*, which are given by a volume integral
$$\int_{V_t} \varrho(x,t) g(x,t) \cdot dx$$
(e.g. gravity or other "external" forces, including sources and sinks)

2. *Surface forces*, which are given by a surface integral
$$\int_{\partial V_t} \sigma_j(x,t) \cdot n(x,t) \cdot dS$$
where $\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$ is the (symmetric) *stress tensor* (e.g. pressure, viscous forces or other "internal" forces)

We can express the surface forces by a volume integral (using Gauss' divergence theorem), and after plugging the sum of forces into equation (1) (which holds for *any* volume V_0), we see that already the integrands must agree; that is:

$$\frac{\partial (\varrho u)}{\partial t} + (\operatorname{div}(\varrho u_j u))_j = \varrho g + \operatorname{div}(\sigma)$$

where $\operatorname{div}(\sigma)$ is defined row-wise.

From the identity

$$\begin{aligned}\operatorname{div}(u_j u) &= \sum_k \partial_k (u_j u_k) = \sum_k (\partial_k u_j) u_k + u_j \sum_k \partial_k u_k \\ &= ((u \cdot \nabla) u)_j + u_j \operatorname{div}(u)\end{aligned}$$

we see that for an incompressible, homogeneous fluid (which is divergence free) this reduces to:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \frac{1}{\varrho_0} \operatorname{div}(\sigma) = g \qquad (1)$$

This is the *equation of motion* of the velocity field.

Now from physics we know that the stress tensor can be decomposed into a pressure part and a viscosity part:

$$\sigma = \underbrace{-pI}_{\text{pressure}} + \underbrace{\tau(\nabla u)}_{\text{viscosity}}$$

($\tau : \Omega \times [0,T] \to \mathbb{R}$ is a scalar function.)

For *inviscid* fluids we have $\tau = 0$, and the equation of motion (1) becomes

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \frac{1}{\varrho_0} \nabla p = g$$

This is called the *Euler equation* for incompressible, homogeneous, inviscid fluids.

For *Newtonian* fluids we only know that $\tau \nabla u$ is linear in ∇u, rotationally invariant and symmetric. This leads to the approximation

$$\tau \nabla u = \lambda \underbrace{\operatorname{div}(u)}_{=0} I + \mu \frac{1}{2} \left(\nabla u + (\nabla u)^T \right)$$

(μ is called *dynamic viscosity* of the fluid). A quick calculation shows that $\operatorname{div}((\nabla u)^T) = 0$, and we arrive at the *Navier-Stokes equation* for incompressible and homogeneous fluids:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \frac{1}{\varrho_0} \nabla p - \nu \Delta u = g$$

(ν is called *kinematic viscosity* of the fluid)